Representations of Erdos spaces by homeomorphism groups and by lower semi-continuous functions on product spaces

Visser, D.

2009

Representations of Erdős spaces by homeomorphism groups and by lower semi-continuous functions on product spaces

Visser, David, 1979
Representations of Erdős spaces by homeomorphism groups and by lower semi-continuous functions on product spaces
ISBN: 978-90-8659-324-8

THOMAS STIELTJES INSTITUTE
FOR MATHEMATICS

NWO
Netherlands Organisation for Scientific Research

2000 Mathematics Subject Classification: 03E15, 28C10, 54C08, 54F50, 57S05

Cover design by Dennis Roubos.

Printed by Ipskamp Drukkers, The Netherlands.

VRIJE UNIVERSITEIT

Representations of Erdős spaces by homeomorphism groups and by lower semi-continuous functions on product spaces

ACADEMISCH PROEFSCHRIFT

ter verkrijging van de graad Doctor aan
de Vrije Universiteit Amsterdam,
op gezag van de rector magnificus
prof.dr. L.M. Bouter,
in het openbaar te verdedigen
ten overstaan van de promotiecommissie
van de faculteit der Exacte Wetenschappen
op maandag 22 juni 2009 om 10.45 uur
in de aula van de universiteit,
De Boelelaan 1105

door

David Visser

geboren te Velsen

promotor: prof.dr. J. J. Dijkstra

Preface

The past four years I have been a PhD student at the VU University Amsterdam and the research done in this period is collected in this thesis. Of course, this thesis would not be possible to write without the help of a number of people and I would like to thank some of them here. First there is my advisor Jan Dijkstra. Jan, thank you for all your help and excellent guidance. I especially appreciate the fact that despite your numerous tasks you always had time for my questions. In addition, your mathematical knowledge and intuition are impressive and it was a pleasure to work with you.

Secondly, I want to thank Jan van Mill. Jan, your topology lectures were inspiring and it is during these lectures that my interest in topology began. I am very thankful that you gave me the opportunity to become a PhD student, working on a project that was funded by the Netherlands Organisation for Scientific Research (NWO).

Then there are some (former) colleagues at the VU University whom I would like to thank. First there is Guit-Jan Ridderbos. Guit-Jan, I really enjoyed our time as roommates and I want to thank you for your help with LaTeX when I was writing this thesis. I am also grateful that you were willing to participate in the reading committee. Next there is Kirsten Valkenburg. Kirsten, it was great working with you and having you as a roommate. Fortunately, we left our room once in a while: our trips to Chicago, Kyoto, and Tokyo are unforgettable! Other people who made my time at the VU University very enjoyable are Mohammad Abry, Esmaeel Asadi, Désirée Basile, Tejaswi Navilarekallu, and Wijnand Rekers. I also want to thank everyone from the OBP-lunch group and all members of the Geometry section, especially Dennis Roubos who helped me with the cover of the thesis and kept me busy with tennis, kart racing, ice skating, making music, and what not.

I am indebted to all members of the reading committee. This committee consists of J. M. Aarts, K. P. Hart, J. van Mill, D. Notbohm, L. Oversteegen, and G. J. Ridderbos. I especially thank Klaas Pieter Hart and Guit-Jan Ridderbos for their careful reading of the entire manuscript and their valuable comments and suggestions.

Last, but not least, I want to thank my family and in particular my parents for all the support they have given me through the years.

Contents

Introduction

The two spaces that form the core of this thesis are *Erdős space*, which we denote by $\mathfrak{E}$, and *complete Erdős space*, which we denote by $\mathfrak{E}_c$. Both spaces were introduced by Erdős [20] in 1940. Erdős space is the space of all sequences of rational numbers in ℓ^2, the Hilbert space of square summable real sequences. Complete Erdős space is the space of all sequences in ℓ^2, every coordinate of which is a point in the convergent sequence $\{0\} \cup \{1/n : n \in \mathbb{N}\}$. Erdős [20] proved that both $\mathfrak{E}$ and $\mathfrak{E}_c$ are one dimensional, yet totally disconnected spaces. Moreover, it is easy to see that $\mathfrak{E}$ and $\mathfrak{E}_c$ are homeomorphic to their own squares, that is, $\mathfrak{E}$ is homeomorphic to $\mathfrak{E}^2$ and $\mathfrak{E}_c$ is homeomorphic to $\mathfrak{E}_c^2$. This means that $\dim \mathfrak{E} = \dim \mathfrak{E}^2 = 1$, which makes this space an important example in dimension theory. Of course, we have a similar situation for $\mathfrak{E}_c$. The spaces $\mathfrak{E}$, $\mathfrak{E}_c$, and also the countable infinite product of $\mathfrak{E}_c$ were topologically characterized by Dijkstra and van Mill [9, 11, 10] and Dijkstra [8].

As will become clear when reading this thesis, $\mathfrak{E}$ and $\mathfrak{E}_c$ appear in many different situations. For example, in Chapter 2 our main result is Theorem 2.4.1, which states that certain homeomorphism groups of n-dimensional Sierpiński carpets for $n \neq 3$ turn out to be homeomorphic to $\mathfrak{E}$. The proof of this result is based on Dijkstra and van Mill [11, Theorem 10.4] where they establish a similar theorem for Menger manifolds. The key step there is the use of one of their topological characterizations of $\mathfrak{E}$, derived in the same article, which is a deep result. Moreover, we heavily use Dijkstra [7, §5] where it is proved that there are closed imbeddings of $\mathfrak{E}_c$ in the homeomorphism group of n-dimensional Sierpiński carpets if $n \neq 3$. This amount of heavy machinery needed to prove Theorem 2.4.1 makes it the main result of this thesis. The reason for throwing the reader immediately into the deep end of the pool like this is that the other chapters deal with various generalizations of known constructions of $\mathfrak{E}$ and $\mathfrak{E}_c$; in Chapter 4 we even generalize the construction of $\mathfrak{E}_c$ to a nonseparable setting. We think it is more convenient to begin with a 'pure' construction of $\mathfrak{E}$, rather than starting with generalized constructions. Chapter 2 is based on Dijkstra and Visser [17].

In Chapter 3 we will proceed with the introduction of generalized Erdős type spaces. The main theorem here is Theorem 3.4.7. This theorem generalizes a result of Dijkstra [5], stated in Theorem 1.2.5, about Erdős type subspaces of ℓ^p, and a result of Dijkstra and van Mill [10], stated in Theorem 3.1.2, about Polishable ideals on the natural numbers. Indeed, at first sight these two subjects are very dissimilar. The space studied in Theorem 3.4.7 is our so-called generalized Erdős space: it generalizes the Erdős type spaces in ℓ^p of Dijkstra and the Polishable ideals on the natural numbers studied by Dijkstra and van Mill. Using a topological characterization of $\mathfrak{E}_c$ by Dijkstra and van Mill [10] we use this theorem to derive some conditions under which this generalized Erdős space is actually homeomorphic to $\mathfrak{E}_c$ in §3.5. In §3.6 we use a topological characterization of $\mathfrak{E}$ by Dijkstra and van Mill [11] to give conditions under which our generalized space is homeomorphic to $\mathfrak{E}$. Finally, we prove a fixed point property in §3.7 that generalizes a result of Abry, Dijkstra and van Mill [2] for the ℓ^p-case. This chapter is based on Dijkstra and Visser [16].

In Chapter 4 we delve into the world of nonseparable spaces. This is motivated by a result of Dijkstra, van Mill and Valkenburg [13]. Whereas in Chapter 3 we generalize Dijkstra's theorem about Erdős type subspaces of ℓ^p by taking a more general function than the norm function on a more general (separable metric) product space than the countable infinite product of the real line, Dijkstra, van Mill and Valkenburg generalize this result in another way. Extending the norm function to an uncountable product of the real line, they were able to derive a theorem similar to that of Dijkstra in this new setting. Using this generalization and topological characterizations of $\mathfrak{E}_c$ by Dijkstra and van Mill [10] they were able to characterize when their resulting, possibly nonseparable, Erdős type spaces are homeomorphic to a so-called nonseparable complete Erdős space. Inspired by these results we extend the theorem of Dijkstra and van Mill about Polishable ideals on the natural numbers, to submeasures on uncountable cardinal numbers. This is the content of Theorem 4.1.2. We are particularly interested in the question when the related ideals are homeomorphic to a nonseparable complete Erdős space. In the last section of Chapter 4 we give a partial answer to this question by showing that for a special class of submeasures the related ideals are homeomorphic to a nonseparable complete Erdős space if and only if the small inductive dimension of these ideals is greater than zero. This chapter is based on Dijkstra, Valkenburg and Visser [15].

In Chapter 1 we discuss the basic theory that is needed for the other chapters.

Chapter 1

Preliminaries

In this chapter we present the basic theory that plays a role in this thesis. All undefined notions can be found in ENGELKING [18, 19]. Except for this chapter and Chapter 4 we will only consider separable metrizable spaces.

1.1 Basic topology

Topology is the study of *topological spaces*. For convenience we will mostly speak of '*space*' instead of 'topological space'. First we introduce some notation for a few well known sets. By ω we denote the set of natural numbers including zero, $\mathbb{N}$ is the set $\omega \setminus \{0\}$, $\mathbb{Z}$ is the set of integers, $\mathbb{Q}$ is the set of rational numbers, and $\mathbb{R}$ is the set of real numbers. We can make these sets into topological spaces by equipping them with the topology induced by the Euclidean metric on $\mathbb{R}$. Unless stated otherwise this will be the topology on these sets when viewed as spaces.

Let A be a subset of a topological space X. We will write $\mathrm{Int}(A)$ for the *interior* of A, the largest open set in X contained in A. We write $\overline{A}$ for the *closure* of the set A, the smallest closed set in X that contains A. Furthermore, we let ∂A be the *boundary* of A, that is the set $\overline{A} \setminus \mathrm{Int}(A)$.

A space X is called *compact* if every open cover of X has a finite subcover. A weaker property is that of *local compactness*, which means that every point of X has a neighbourhood U with compact closure $\overline{U}$.

A space X is called *separable* if it has a countable dense subset. It is *metrizable* if there is a metric on X that generates the topology of X. Such a metric is called *compatible* or *admissible*. Note that all of the spaces ω, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ mentioned above are separable and metrizable.

Let d be an admissible metric for a space X. A sequence (x_n) of elements of X is a *Cauchy sequence* if for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $d(x_m, x_n) \leq \varepsilon$ for all $n, m \geq N$. We call the metric space (X, d) *complete* if every Cauchy sequence has a limit in X. In this case we say that the space X is *completely metrizable*. A separable completely metrizable space is called *Polish*.

Let d_1 and d_2 be two metrics on a set X. We say that d_1 and d_2 are *uniformly equivalent* if for every $\varepsilon > 0$ there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that for all $x, x' \in X$ we have $d_2(x, x') < \varepsilon$ whenever $d_1(x, x') < \delta_1$ and $d_1(x, x') < \varepsilon$ whenever $d_2(x, x') < \delta_2$. For a metric space (X, d) and $A \subset X$ we define the *diameter* of A by $\operatorname{diam} A = \sup\{d(x, y) : x, y \in A\}$.

1.1.1 Homeomorphism groups

In Chapter 2 we will study certain homeomorphism groups so let us recall the relevant notions involving this subject.

A group $(G, \cdot)$ which is also a topological space is a *topological group* if the function $G \times G \to G$ defined by

$$(x, y) \mapsto x \cdot y^{-1}$$

is continuous. Let X be a space. We let $\mathcal{H}(X)$ denote the *autohomeomorphism group* of X. The group operation here is of course the composition operation. If $A, B \subset X$ then we define $[A, B] = \{h \in \mathcal{H}(X) : h(A) \subset B\}$. The collection of sets of the form $[K, O]$ where $K \subset X$ is compact and $O \subset X$ is open forms a subbasis for the *compact-open topology* on $\mathcal{H}(X)$. We will only consider the space $\mathcal{H}(X)$ for locally compact spaces X. If X is compact then the compact-open topology coincides with the topology of uniform convergence with respect to any compatible metric for X. Moreover, in this case the compact-open topology makes $\mathcal{H}(X)$ into a topological group that is a Polish space. If X is locally compact but not compact the situation is more complex. In this case, the topology of uniform convergence depends on the metric that one chooses for X and it is usually much stronger than the compact-open topology. Furthermore, the compact-open topology is not necessarily compatible with the group structure of $\mathcal{H}(X)$ anymore, see e.g. DIJKSTRA [6]. We write $X \approx Y$ to denote the fact that X is homeomorphic to the space Y.

1.1.2 (Sub)measures and ideals

Let A be an arbitrary set and let $\mathcal{P}(A)$ be the *powerset* of A, that is, the collection of all subsets of A. We state the definition of a *submeasure* on A.

Definition 1.1.1. A *submeasure* φ on A is a function $\varphi : \mathcal{P}(A) \to [0, \infty]$ such that

(a) $\varphi(\emptyset) = 0$;

(b) $0 < \varphi(\{x\}) < \infty$ for any point $x \in A$;

(c) (*monotonicity*) $\varphi(X) \leq \varphi(Y)$ for all $X \subset Y \subset A$; and

(d) (*subadditivity*) $\varphi(X \cup Y) \leq \varphi(X) + \varphi(Y)$ for all $X, Y \subset A$.

Example 1.1.2. The function $\varphi \colon \mathcal{P}(\omega) \to \{0, 1\}$ given by

$$\varphi(X) = \begin{cases} 0, & \text{if } X = \emptyset; \\ 1, & \text{if } X \neq \emptyset. \end{cases}$$

is an example of a submeasure on ω.

A special class of submeasures is formed by the so-called *measures*.

Definition 1.1.3. A *measure* φ on A is a function $\varphi : \mathcal{P}(A) \to [0, \infty]$ such that

(a) $\varphi(\emptyset) = 0$;

(b) $0 < \varphi(\{x\}) < \infty$ for any point $x \in A$; and

(c) (*additivity*) $\varphi(X \cup Y) = \varphi(X) + \varphi(Y)$ for any two disjoint subsets $X, Y \subset A$.

Example 1.1.4. The function $\varphi\colon \mathcal{P}(\omega) \to [0,\infty]$ given by

$$\varphi(X) = \sum_{n\in X} \frac{1}{n+1}$$

is a measure on ω, where we use the convention that the empty sum is equal to zero.

Associated with a submeasure φ on A we introduce the following collections of subsets of A:

$$\begin{aligned}\mathrm{Exh}(\varphi) = \{X \subset A : {}&\text{for all } \varepsilon > 0 \text{ there is a finite subset } F \text{ of } A\\ &\qquad\qquad\text{such that } \varphi(X \setminus F) < \varepsilon\},\end{aligned}$$

and

$$\mathrm{Fin}(\varphi) = \{X \subset A : \varphi(X) < \infty\}.$$

We have the following simple result.

Lemma 1.1.5. *Let φ be a submeasure on a set A. Then* $\mathrm{Exh}(\varphi) \subset \mathrm{Fin}(\varphi)$.

Proof. Let $X \in \mathrm{Exh}(\varphi)$. By definition of $\mathrm{Exh}(\varphi)$ we can find a finite set F in A such that $\varphi(X \setminus F) < 1$. Note that $X \subset (X \setminus F) \cup F$, so with the subadditivity of φ it follows that $\varphi(X) \leq \varphi(X \setminus F) + \varphi(F) < 1 + \varphi(F)$. Again with the subadditivity of φ we see that $\varphi(F) \leq \sum_{x\in F} \varphi(\{x\})$, which is finite by property (b) of Definition 1.1.1. We conclude that $\varphi(X) < \infty$. □

We denote the cardinality of A by $|A|$. If $|A| < \omega$ it is easy to see that $\mathrm{Exh}(\varphi) = \mathrm{Fin}(\varphi) = \mathcal{P}(A)$. In the next lemma we prove that in general every element of $\mathrm{Exh}(\varphi)$ is at most countable. Another proof of this fact can be found in Lemma 4.3.1.

Lemma 1.1.6. *Let φ be a submeasure on a set A. Then every element of* $\mathrm{Exh}(\varphi)$ *is at most countable.*

PROOF. Take $X \in \mathrm{Exh}(\varphi)$ and let $X_n = \{x \in X : \varphi(\{x\}) > 1/n\}$ for $n \in \mathbb{N}$. Note that $X = \bigcup_{n=1}^{\infty} X_n$ and that $X_n \in \mathrm{Exh}(\varphi)$ for all $n \in \mathbb{N}$. Take $m \in \mathbb{N}$. We can find a finite subset F of X_m such that $\varphi(X_m \setminus F) < 1/m$. This implies that $\varphi(\{x\}) < 1/m$ and $\varphi(\{x\}) > 1/m$ for $x \in X_m \setminus F$, so $X_m \setminus F = \emptyset$. We conclude that X_m is a finite set and hence X is at most countable. $\square$

Note that for $A = \omega$ the set $\mathrm{Exh}(\varphi)$ is equal to the set

$$\{A \subset \omega : \lim_{m\to\infty} \varphi(A \setminus m) = 0\}, \tag{1.1}$$

where a number $m \in \omega$ in a set theoretic context stands for the empty set if $m = 0$ and for the set $\{0, \ldots, m-1\}$ if $m \in \mathbb{N}$. Furthermore, it is easily verified that $\mathrm{Exh}(\varphi)$ and $\mathrm{Fin}(\varphi)$ are *ideals* on A. We recall the definition of an ideal.

Definition 1.1.7. An *ideal* $\mathcal{I}$ on A is a subset of $\mathcal{P}(A)$ such that $\mathcal{I}$ contains the finite sets, $X \in \mathcal{I}$ whenever $X \subset Y \in \mathcal{I}$, and $X \cup Y \in \mathcal{I}$ whenever $X, Y \in \mathcal{I}$.

The ideal $\mathrm{Exh}(\varphi)$ is called the *exhaustive ideal of* φ and $\mathrm{Fin}(\varphi)$ is called the *finite ideal of* φ. For example, for the submeasure φ of Example 1.1.2 we have $\mathrm{Exh}(\varphi) = \{X \subset \omega : X \text{ is finite}\}$ and for φ in Example 1.1.4 we see that $\mathrm{Exh}(\varphi) = \mathrm{Fin}(\varphi) = \{X \subset \omega : \sum_{n\in X} 1/(n+1) < \infty\}$.

We can make the set $\mathcal{P}(A)$ into a topological space by equipping it with the standard product topology that comes with identification with 2^A. If $X \in \mathcal{P}(A)$ then $\{Y \subset A : Y \cap F = X \cap F\}$ for some finite set $F \subset A$ is a standard neighbourhood of X in $\mathcal{P}(A)$. Note that for uncountable sets A the resulting space is not metrizable and if $|A| > \mathfrak{c}$, where $\mathfrak{c}$ denotes the cardinality of $\mathbb{R}$, this space is also not separable. If we take the symmetric difference '$\triangle$' as the group operation on $\mathcal{P}(A)$ we have made $\mathcal{P}(A)$ into a topological group.

Now that we have a topology on $\mathcal{P}(A)$ we can speak of *lower semi-continuous* submeasures. We state the definition of lower semi-continuity for arbitrary functions. We write $\hat{\mathbb{R}}$ for the compactification $[-\infty, \infty]$ of $\mathbb{R}$.

Definition 1.1.8. Let Z be a space. A function $f\colon Z \to \hat{\mathbb{R}}$ is called *lower semi-continuous* (abbreviated LSC) if $f^{-1}\big((t,\infty]\big)$ is open in Z for every $t \in \mathbb{R}$.

It is easily checked that for a first countable space space Z the function $f\colon Z \to \hat{\mathbb{R}}$ is LSC if and only if for every convergent sequence $(z_n)_{n\in\omega}$ in Z we have $f(\lim_{n\to\infty} z_n) \le \liminf_{n\to\infty} f(z_n)$. When we speak of an LSC submeasure φ on A we mean that φ as a function from $\mathcal{P}(A)$ to $[0,\infty]$ is LSC with respect to the product topology on $\mathcal{P}(A)$. We will also (more correctly) say that φ is LSC on $\mathcal{P}(A)$ in this case. We introduce some notation. If $f\colon W \to Z$ is a function and $V \subset W$ we denote by $f{\restriction}V$ the restriction of f to V. It is not difficult to show that every submeasure φ is an LSC function on $\mathrm{Exh}(\varphi)$, by which we mean that $\varphi{\restriction}\mathrm{Exh}(\varphi)\colon \mathrm{Exh}(\varphi) \to [0,\infty)$ is an LSC function with respect to the topology that $\mathrm{Exh}(\varphi)$ inherits from $\mathcal{P}(A)$. We shall simply refer to this topology as the 'product topology'.

Lemma 1.1.9. *Let φ be a submeasure on a set A. Then φ is an LSC function on* $\mathrm{Exh}(\varphi)$.

Proof. Take $t \in [0,\infty)$. We have to show that $U = \{X \in \mathrm{Exh}(\varphi) : \varphi(X) > t\}$ is open in $\mathrm{Exh}(\varphi)$ with the product topology. Let $X \in U$ and put $\delta = \varphi(X) - t$, so $\delta > 0$. By definition of $\mathrm{Exh}(\varphi)$ we can find a finite subset F of X such that $\varphi(X \setminus F) < \delta$. With the subadditivity of φ it follows that

$$\begin{aligned}\varphi(X \cap F) &\ge \varphi(X) - \varphi(X \setminus F)\\ &> \varphi(X) - \delta\\ &= t.\end{aligned}$$

Consider the neighbourhood $V = \{Y \in \mathrm{Exh}(\varphi) : Y \cap F = X \cap F\}$ of X in $\mathrm{Exh}(\varphi)$. It follows from the monotonicity of φ that for all $Y \in V$ we have $\varphi(Y) \ge \varphi(Y \cap F) = \varphi(X \cap F) > t$. This means that $X \in V \subset U$ and we conclude that U is indeed open in $\mathrm{Exh}(\varphi)$. $\square$

If $|A| > \omega$ and $(x_\alpha)_{\alpha\in A}$ is a sequence of nonnegative numbers we define the sum of all the numbers x_α, in analogy with the definition of countable infinite sums, as

$$\sum_{\alpha \in A} x_\alpha = \sup \left\{ \sum_{\alpha \in I} x_\alpha : I \subset A \text{ and } |I| < \omega \right\}.$$

We make the following observation.

Lemma 1.1.10. *Let φ be an LSC submeasure on a set A. Then we have*

$$\sup_{x \in X} \varphi(\{x\}) \leq \varphi(X) \leq \sum_{x \in X} \varphi(\{x\})$$

for all $X \subset A$.

Proof. Take $X \subset A$. It follows immediately from the monotonicity of φ that $\sup_{x \in X} \varphi(\{x\}) \leq \varphi(X)$, so it is left to show that $\varphi(X) \leq \sum_{x \in X} \varphi(\{x\})$. If $\sum_{x \in X} \varphi(\{x\}) = \infty$ there is nothing to prove, so suppose that $\sum_{x \in X} \varphi(\{x\}) < \infty$. This means that $|X| \leq \omega$ by property (b) of Definition 1.1.1. If $|X| < \omega$, then it follows easily from the subadditivity of φ that $\varphi(X) \leq \sum_{x \in X} \varphi(\{x\})$. If $|X| = \omega$ we can find an enumeration $X = \{x_1, x_2, \ldots\}$. Define for $n \in \mathbb{N}$ the set $X_n \subset X$ as $X_n = \{x_1, \ldots, x_n\}$. Note that $\lim_{n \to \infty} X_n = X$ in $\mathcal{P}(A)$. The lower semi-continuity and monotonicity of φ imply that $\varphi(X) = \lim_{n \to \infty} \varphi(X_n)$. The result now follows from the subadditivity of φ. $\square$

For LSC measures we can derive a stronger result.

Lemma 1.1.11. *Let φ be an LSC measure on a set A. Then we have*

$$\sup_{x \in X} \varphi(\{x\}) \leq \varphi(X) = \sum_{x \in X} \varphi(\{x\})$$

for all $X \subset A$.

Proof. Suppose that $\sum_{x \in X} \varphi(\{x\}) = \infty$. We show that in this case $\varphi(X) = \infty$. The lemma then follows from the proof of Lemma 1.1.10, with the difference that we can use the additivity of φ to get equalities rather than inequalities. Take an $n \in \mathbb{N}$. We can find a finite set $F \subset X$ such that $\varphi(F) = \sum_{x \in F} \varphi(\{x\}) > n$. Since φ is monotone we see that $\varphi(X) \geq \varphi(F) > n$. We conclude that $\varphi(X) = \infty$. $\square$

Together with Lemma 1.1.5 this lemma implies the following result.

Lemma 1.1.12. *Let φ be an LSC measure on a set A. Then we have*

$$\mathrm{Exh}(\varphi) = \mathrm{Fin}(\varphi).$$

1.1.3 Dimension theory

We briefly discuss some notions from dimension theory. The main reason for this is that in Chapter 4 we study submeasures on uncountable cardinals where we equip the corresponding exhaustive ideals with a metric topology that is no longer separable. For nonseparable metric spaces the three dimension functions dim, ind and Ind no longer coincide as they do for separable metrizable spaces. Our main references for dimension theory are Chapter 7 of Engelking [19] and Engelking [18]. In particular the definitions of the covering dimension dim, the small inductive dimension ind and the large inductive dimension Ind can be found there. The notion of dimension that we will use for the nonseparable metric ideals in Chapter 4 is the small inductive dimension ind. For completeness sake we state the definition of ind here.

Definition 1.1.13. Let X be a regular space and let $n \in \omega$. We say that

$\operatorname{ind} X = -1$	$\Leftrightarrow$	$X = \emptyset$;
$\operatorname{ind} X \leq n$	$\Leftrightarrow$	for every $x \in X$ and every neighbourhood U of x there is an open neighbourhood V of x with $V \subset U$ such that $\operatorname{ind} \partial V \leq n-1$;
$\operatorname{ind} X = n$	$\Leftrightarrow$	$\operatorname{ind} X \leq n$ and $\operatorname{ind} X \not\leq n-1$;
$\operatorname{ind} X = \infty$	$\Leftrightarrow$	$\operatorname{ind} X \not\leq n$ for any n.

The small inductive dimension ind is also called the *Menger-Urysohn dimension*. The reason that we choose the dimension ind for the nonseparable metric ideals in Chapter 4 is that we are interested in the question whether these spaces have a basis consisting of clopen sets. For a nonempty space X this question is easily seen to be equivalent with asking whether $\operatorname{ind} X = 0$.

Definition 1.1.14. A regular space X is called *zero-dimensional* if $\operatorname{ind} X = 0$.

1.2 Erdős spaces and almost zero-dimensionality

In this section we assume all spaces to be separable and metrizable. Remember that $\hat{\mathbb{R}}$ denotes the compactification $[-\infty, \infty]$ of $\mathbb{R}$. The two most important spaces in this thesis are *Erdős space* $\mathfrak{E}$ and *complete Erdős space* $\mathfrak{E}_c$. We introduce these spaces in ℓ^2 but we will consider ℓ^p-spaces for an arbitrary real number $p \geq 1$ as well. For such a number p the Banach space ℓ^p consists of all sequences $z = (z_0, z_1, \ldots) \in \mathbb{R}^\omega$ such that $\sum_{n=0}^{\infty} |z_n|^p < \infty$. The topology on ℓ^p is generated by the p-norm $\|z\|_p = \left(\sum_{n=0}^{\infty} |z_n|^p\right)^{1/p}$.

We have the following important result about convergence in ℓ^p.

Proposition 1.2.1. *Let $p \geq 1$ and suppose that $\big(x(n)\big)_n$ is a sequence in ℓ^p and $x \in \ell^p$. Then the following statements are equivalent:*

(a) $\lim_{n\to\infty} x(n) = x$ *in ℓ^p;*

(b) $\lim_{n\to\infty} \|x(n)\|_p = \|x\|_p$ *and for every $i \in \omega$,* $\lim_{n\to\infty} x(n)_i = x_i$.

Proof. First we show that (a) $\Rightarrow$ (b). The triangle inequality for $\|\cdot\|_p$ directly implies that $\big|\|x(n)\|_p - \|x\|_p\big| \leq \|x(n) - x\|_p$, from which it follows that $\lim_{n\to\infty} \|x(n)\|_p = \|x\|_p$. That $\lim_{n\to\infty} x(n)_i = x_i$ for every $i \in \omega$ is a triviality.

Now we prove that (b) $\Rightarrow$ (a). Let $\varepsilon > 0$. As $x \in \ell^p$ we can find an $m \in \omega$ such that

$$\sum_{i=m+1}^{\infty} |x_i|^p < \frac{1}{8}\left(\frac{\varepsilon}{2}\right)^p. \tag{1.2}$$

Furthermore, we can choose an $N \in \omega$ such that for every $n \geq N$ we have the following three inequalities:

$$\begin{aligned} \Big|\|x(n)\|_p^p - \|x\|_p^p\Big| &< \frac{1}{8}\left(\frac{\varepsilon}{2}\right)^p, && (1.3)\\ \sum_{i=0}^{m} |x(n)_i - x_i|^p &< \frac{\varepsilon^p}{2}, \text{ and} && (1.4)\\ \sum_{i=0}^{m}\Big||x(n)_i|^p - |x_i|^p\Big| &< \frac{1}{8}\left(\frac{\varepsilon}{2}\right)^p. && (1.5)\end{aligned}$$

Since

$$\sum_{i=m+1}^{\infty} |x(n)_i|^p - \sum_{i=m+1}^{\infty} |x_i|^p = \|x(n)\|_p^p - \sum_{i=0}^{m} |x(n)_i|^p + \sum_{i=0}^{m} |x_i|^p - \|x\|_p^p,$$

we find with the triangle inequality, (1.3), and (1.5) that for all $n \geq N$ we have

$$\begin{aligned} \left| \sum_{i=m+1}^{\infty} |x(n)_i|^p - \sum_{i=m+1}^{\infty} |x_i|^p \right| &< \frac{1}{8} \left(\frac{\varepsilon}{2}\right)^p + \sum_{i=0}^{m} \Big| |x(n)_i|^p - |x_i|^p \Big| \\ &< \frac{1}{4} \left(\frac{\varepsilon}{2}\right)^p. \end{aligned}$$

This implies with (1.2) that

$$\sum_{i=m+1}^{\infty} |x(n)_i|^p < \frac{3}{8} \left(\frac{\varepsilon}{2}\right)^p$$

for all $n \geq N$.

Using this inequality together with (1.2) and (1.4) we find for every $n \geq N$ that

$$\begin{aligned} \|x(n) - x\|_p^p &= \sum_{i=0}^{m} |x(n)_i - x_i|^p + \sum_{i=m+1}^{\infty} |x(n)_i - x_i|^p \\ &\leq \sum_{i=0}^{m} |x(n)_i - x_i|^p + \sum_{i=m+1}^{\infty} \big(2 \max\{|x(n)_i|, |x_i|\}\big)^p \\ &\leq \sum_{i=0}^{m} |x(n)_i - x_i|^p + 2^p \sum_{i=m+1}^{\infty} |x(n)_i|^p + 2^p \sum_{i=m+1}^{\infty} |x_i|^p \\ &< \frac{\varepsilon^p}{2} + \frac{3\,\varepsilon^p}{8} + \frac{\varepsilon^p}{8} = \varepsilon^p \end{aligned}$$

and hence $\|x(n) - x\|_p < \varepsilon$ if $n \geq N$. □

This proposition shows that the norm topology on ℓ^p is the weakest topology that contains the product topology inherited from $\mathbb{R}^\omega$ and that makes the norm function continuous. Since the product topology is generated by the coordinate projections $x \mapsto x_n$, another way to say this is that the norm topology on ℓ^p is the weakest topology that makes all the coordinate projections on ℓ^p and the norm function continuous. This means precisely that $\|\cdot\|_p$ is a *Kadec norm for the coordinate projections on* ℓ^p: a norm $|\cdot|$ on ℓ^p is called a Kadec norm for the coordinate projections on ℓ^p if it is topologically equivalent to $\|\cdot\|_p$ and if it has the property that the norm topology on ℓ^p is the weakest topology that makes all the coordinate projections and the function $|\cdot|$ continuous, see Bessaga and Pełczyński [3, Chapter VI, §3]. We extend the p-norm over $\hat{\mathbb{R}}^\omega$ by putting $\|z\|_p = \infty$ for each $z \in \hat{\mathbb{R}}^\omega \setminus \ell^p$. Observe that the norm as a function from $\hat{\mathbb{R}}^\omega$ to $[0, \infty]$ is not continuous because the norm topology on ℓ^p is much stronger than the product topology on ℓ^p. The norm function is however an LSC function on $\hat{\mathbb{R}}^\omega$.

Lemma 1.2.2. *Let $p \geq 1$. Every closed ball $\{x \in \ell^p : \|x\|_p \leq t\}$ for $t > 0$ is a closed subset of $\hat{\mathbb{R}}^\omega$.*

Proof. Take $p \geq 1$. Let $t > 0$ and suppose that $x \in \hat{\mathbb{R}}^\omega$ is such that $\|x\|_p > t$. Then we can find an $m \in \omega$ such that $\sum_{i=0}^m |x_i|^p > t^p$. Since this sum is a continuous function of the vector $(x_0, \ldots, x_m)$ we can find a $\delta > 0$ such that $\sum_{i=0}^m |y_i|^p > t^p$ whenever $|x_i - y_i| < \delta$ for $0 \leq i \leq m$. Consider the basic open neighbourhood U of x with respect to the product topology on $\hat{\mathbb{R}}^\omega$ given by

$$U = \{y \in \hat{\mathbb{R}}^\omega : |x_i - y_i| < \delta \text{ for } 0 \leq i \leq m\}.$$

Clearly, $\|y\|_p > t$ for all $y \in U$. □

With Proposition 1.2.1 and Lemma 1.2.2 we see that we can also describe the norm topology on ℓ^p as the topology that is generated by the product topology together with the sets $\{z \in \ell^p : \|z\|_p < t\}$ for $t > 0$. We note here that Bessaga and Pełczyński [3, p.176] showed that the lower semi-continuity of $\|\cdot\|_p$ with respect to the product topology on ℓ^p already follows from the fact that the p-norm is a Kadec norm for the coordinate projections on ℓ^p.

We define *Erdős space*

$$\mathfrak{E} = \{x \in \ell^2 : x_n \in \mathbb{Q} \text{ for all } n \in \omega\},$$

and *complete Erdős space*

$$\mathfrak{E}_{\mathrm{c}} = \{x \in \ell^2 : x_n \in \{0\} \cup \{1/m : m \in \mathbb{N}\} \text{ for all } n \in \omega\}.$$

Let $\mathcal{T}$ be the zero-dimensional topology that $\mathfrak{E}$ inherits from $\mathbb{Q}^\omega$. We noted that $\mathcal{T}$ is weaker than the norm topology so we have that the clopen sets separate the points of $\mathfrak{E}$, that is, $\mathfrak{E}$ is totally disconnected. It follows from Proposition 1.2.1 that the graph of the norm function, when seen as a function from $(\mathfrak{E}, \mathcal{T})$ to $\mathbb{R}^+$, is homeomorphic to $\mathfrak{E}$. This means that we can informally think of $\mathfrak{E}$ as a 'zero-dimensional space with some LSC function declared continuous'. Of course, the same holds true for $\mathfrak{E}_{\mathrm{c}}$.

We point out the following connection between the two topologies on $\mathfrak{E}$. It follows from Lemma 1.2.2 that every closed ε-ball in $\mathfrak{E}$ is also closed in the zero-dimensional space $\mathbb{Q}^\omega$. This means that every point in $\mathfrak{E}$ has arbitrarily small neighbourhoods which are intersections of clopen sets. Clearly, Lemma 1.2.2 also implies that every closed ε-ball in $\mathfrak{E}_{\mathrm{c}}$ is also closed in the zero-dimensional space $(\{0\} \cup \{1/m : m \in \mathbb{N}\})^\omega$.

Definition 1.2.3. A subset A of a space X is called a *C-set* in X if A can be written as an intersection of clopen subsets of X. A space is called *almost zero-dimensional* if every point of the space has a neighbourhood basis consisting of C-sets of the space. If Z is a set that contains X then we say that a (separable metric) topology $\mathcal{T}$ on Z *witnesses the almost zero-dimensionality of* X if $\dim(Z, \mathcal{T}) \leq 0$, $O \cap X$ is open in X for each $O \in \mathcal{T}$, and every point of X has a neighbourhood basis in X consisting of sets that are closed in $(Z, \mathcal{T})$. We will also say that the space $(Z, \mathcal{T})$ is a witness to the almost zero-dimensionality of X.

Thus $\mathfrak{E}$ and $\mathfrak{E}_{\mathrm{c}}$ are almost zero-dimensional spaces. The space $\mathbb{Q}^\omega$ is a witness to the almost zero-dimensionality of $\mathfrak{E}$ and the space $(\{0\} \cup \{1/m : m \in \mathbb{N}\})^\omega$ is a witness to the almost zero-dimensionality of $\mathfrak{E}_{\mathrm{c}}$. More generally, if $\varphi\colon Z \to \mathbb{R}$ is an LSC function with a zero-dimensional domain then it follows easily that Z is a witness to the almost zero-dimensionality of the graph of φ. Clearly, a space X is almost zero-dimensional if and only if there is a topology on X witnessing this fact, see [11, Remark 2.4]. Oversteegen and Tymchatyn [26] proved that every almost zero-dimensional space is at most one-dimensional; see also LEVIN and POL [23] and ABRY and DIJKSTRA [1].

In fact, ERDŐS [20] proved that both $\mathfrak{E}$ and $\mathfrak{E}_\mathrm{c}$ are one-dimensional. This result together with the obvious fact that both spaces are homeomorphic to their squares make these spaces important examples in Dimension Theory. The spaces $\mathfrak{E}$, $\mathfrak{E}_\mathrm{c}$, and also $\mathfrak{E}_\mathrm{c}^\omega$ were characterized by DIJKSTRA and VAN MILL [11, 9, 10] and DIJKSTRA [8].

The following definition is of importance for the topological characterizations of $\mathfrak{E}$ and $\mathfrak{E}_\mathrm{c}$. We state it here in view of Theorem 1.2.5, which plays a central role in Chapter 3.

Definition 1.2.4. Let X be a space and let $\mathcal{A}$ be a collection of subsets of X. The space X is called $\mathcal{A}$-*cohesive* if every point of the space has a neighbourhood that does not contain nonempty clopen subsets of any element of $\mathcal{A}$. If a space X is $\{X\}$-cohesive then we simply call X *cohesive*.

Take $p \geq 1$. A subset A of ℓ^p is called *bounded* if it is bounded in norm, that is, if there is an $M \in \mathbb{N}$ such that $\|a\|_p \leq M$ for all $a \in A$. If A is not bounded we call it an *unbounded* set. Note that A is bounded if and only if $\operatorname{diam} A < \infty$. As a generalization of the construction of $\mathfrak{E}$ and $\mathfrak{E}_\mathrm{c}$, consider a fixed sequence $E_0, E_1, E_2, \ldots$ of subsets of $\mathbb{R}$ and let

$$\mathcal{E} = \{z \in \ell^p : z_n \in E_n \text{ for every } n \in \omega\}.$$

In Chapter 3 we will generalize this construction even further and there we will also give a generalization of the following result of DIJKSTRA [5, Theorem 1].

Theorem 1.2.5. *Assume that $\mathcal{E}$ is not empty and that every E_n is zero-dimensional. For each $\varepsilon > 0$ we let $\eta(\varepsilon) \in \mathbb{R}^\omega$ be given by*

$$\eta(\varepsilon)_n = \sup\{|a| : a \in E_n \cap [-\varepsilon, \varepsilon]\},$$

where $\sup \emptyset = 0$. *The following statements are equivalent:*

(1) $\|\eta(\varepsilon)\|_p = \infty$ *for each* $\varepsilon > 0$*;*

(2) *there exists an* $x \in \prod_{n=0}^\infty E_n$ *with* $\|x\|_p = \infty$ *and* $\lim_{n\to\infty} x_n = 0$*;*

(3) *every nonempty clopen subset of* $\mathcal{E}$ *is unbounded;*

(4) $\mathcal{E}$ *is cohesive ; and*

(5) $\dim \mathcal{E} > 0$.

Note that under the conditions of this theorem the space $\mathcal{E}$ is almost zero-dimensional: the product space $\prod_{n=0}^{\infty} E_n$ is a witness to the almost zero-dimensionality of $\mathcal{E}$. Since Oversteegen and Tymchatyn proved that every almost zero-dimensional space is at most one-dimensional, we might as well write $\dim \mathcal{E} = 1$ in condition (5). This can also be derived from Proposition 1.2.1, which states that the graph of the norm function when seen as a function from ℓ^p with the product topology inherited from $\mathbb{R}^\omega$ to $\mathbb{R}$, is homeomorphic to the Banach space ℓ^p. We see then that the norm topology on spheres $S_\varepsilon(a) = \{x \in \ell^p : \|x-a\| = \varepsilon\}$, for $\varepsilon > 0$ and $a \in \ell^p$, coincides with the product topology. This implies that the spheres in $\mathcal{E}$ are zero-dimensional if the sets E_n are zero-dimensional. It follows from Definition 1.1.13 and the fact that $\operatorname{ind} \mathcal{E} = \dim \mathcal{E}$ (see ENGELKING [18, Theorem 1.7.7]) that $\dim \mathcal{E} \leq 1$ in that case.

Recall that if $A_0, A_1, \ldots$ is a sequence of subsets of a space X then $\limsup_{n\to\infty} A_n = \bigcap_{n=0}^{\infty} \overline{\bigcup_{k=n}^{\infty} A_k}$. A point x in a topological space X is called a *cluster point of a set* $A \subset X$ if $x \in \overline{A \setminus \{x\}}$. The following sufficient condition for $\dim \mathcal{E} \neq 0$ by DIJKSTRA [5, Corollary 2] is a useful one because it is easily tested.

Corollary 1.2.6. *If* 0 *is a cluster point of* $\limsup_{n\to\infty} E_n$ *then every nonempty clopen subset of* $\mathcal{E}$ *is unbounded* (*and hence* $\dim \mathcal{E} \neq 0$).

For example, this corollary immediately implies that $\mathfrak{E}$ and $\mathfrak{E}_{\mathrm{c}}$ are cohesive spaces and hence $\dim \mathfrak{E} = \dim \mathfrak{E}_{\mathrm{c}} = 1$ since we already know that they are at most one-dimensional. The following theorem is a result by DIJKSTRA [5, Theorem 3].

Theorem 1.2.7. *If every set* E_n *is closed in* $\mathbb{R}$, *then* $\mathcal{E}$ *is homeomorphic to complete Erdős space if and only if* $\dim \mathcal{E} > 0$ *and every* E_n *is zero-dimensional.*

Consider for example the case that $E_n = \{0, 1/(n+1)\}$ for $n \in \omega$. It follows from Theorem 1.2.5 and the well-known fact that $\sum_{n=0}^{\infty} 1/(n+1) = \infty$ that $\dim \mathcal{E} > 0$, so we can apply Theorem 1.2.7 to see that $\mathcal{E}$ is homeomorphic to $\mathfrak{E}_\mathrm{c}$. Dijkstra [5] refers to this minimal representation of $\mathfrak{E}_\mathrm{c}$ as the *harmonic Erdős space.* In Chapter 3 we give a generalization of Theorem 1.2.7 in Theorem 3.5.3.

1.3 Topological characterizations of Erdős space

Of great importance in this thesis are the aforementioned topological characterizations of $\mathfrak{E}$ that can be found in Dijkstra and van Mill [9, 11]. As these characterizations are deep results we already state them here, so that the reader can get acquainted with them in this early stage. Before we can formulate these characterizations we need to introduce some new notions. The following definitions are taken from Dijkstra and van Mill [11]. *In this section we assume all spaces to be separable and metrizable.*

Definition 1.3.1. If A is a nonempty set then $A^{<\omega}$ denotes the set of all finite strings of elements of A, including the null string λ. If $s \in A^{<\omega}$ then $|s|$ denotes its *length.* In this context the set A is called an *alphabet.* Let A^ω denote the set of all infinite strings of elements of A. If $s \in A^{<\omega}$ and $\sigma \in A^{<\omega} \cup A^\omega$ then we put $s \prec \sigma$ if s is an initial substring of σ, that is, there is a $\tau \in A^{<\omega} \cup A^\omega$ with $s^\frown\tau = \sigma$, where $^\frown$ denotes concatenation of strings. If $\sigma \in A^{<\omega} \cup A^\omega$ and $k \in \omega$ then $\sigma \restriction k \in A^{<\omega}$ is characterized by $\sigma \restriction k \prec \sigma$ and $|\sigma \restriction k| = k$.

Definition 1.3.2. A *tree* T on an alphabet A is a subset of $A^{<\omega}$ that is closed under initial segments, i.e. if $s \in T$ and $t \prec s$ then $t \in T$. An *infinite branch* of T is an element σ of A^ω such that $\sigma \restriction k \in T$ for every $k \in \omega$. The *body* of T, written as $[T]$ is the set of all infinite branches of T. If $s, t \in T$ are such that $s \prec t$ and $|t| = |s| + 1$ then we say that t is an *immediate successor* of s and $\mathrm{succ}(s)$ denotes the set of immediate successors of s in T.

If $(X_n)_{n\in\omega}$ is a sequence of subsets of a space X and $x \in X$, we say that the sequence $(X_n)_n$ converges to x if for every neighbourhood U of x almost all sets X_n are contained in U, that is, there is an $N \in \omega$

such that $X_n \subset U$ for all $n \geq N$. We can now introduce the concept of an *anchor*.

Definition 1.3.3. Let T be a tree and let $(X_s)_{s\in T}$ be a system of subsets of a space X such that $X_t \subset X_s$ whenever $s \prec t$. A subset A of X is called an *anchor* for $(X_s)_{s\in T}$ in X if for every $\sigma \in [T]$ we have $X_{\sigma\restriction k} \cap A = \emptyset$ for some $k \in \omega$ or the sequence $X_{\sigma\restriction 0}, X_{\sigma\restriction 1}, \ldots$ converges to a point in X.

Thus the anchor A has the property that for every sequence that is generated by an element of $[T]$, if it is attached to A, then it must converge to a point in the space.

Example 1.3.4. As noted in §1.2, the space $\mathbb{Q}^\omega$ is a witness to the almost zero-dimensionality of $\mathfrak{E}$. Let $\mathfrak{T}$ be the topology that $\mathfrak{E}$ inherits from $\mathbb{Q}^\omega$. Put $T = \mathbb{Q}^{<\omega}$ and let for $s = q_0 \ldots q_{k-1} \in T$, with $k \in \omega$, the closed subset $\mathbb{Q}^\omega_s$ of $\mathbb{Q}^\omega$ be given by

$$\mathbb{Q}^\omega_s = \{x \in \mathbb{Q}^\omega : x_i = q_i \text{ for } 0 \leq i < k\}.$$

Put $\mathfrak{E}_s = \mathbb{Q}^\omega_s \cap \mathfrak{E}$ for $s \in T$ and let B be a bounded subset of $\mathfrak{E}$. We show that B is an anchor for $(\mathfrak{E}_s)_{s\in T}$ in $(\mathfrak{E}, \mathfrak{T})$. Let $\sigma = (q_0, q_1, \ldots) \in [T]$ be such that $\mathfrak{E}_{\sigma\restriction k} \cap B \neq \emptyset$ for all $k \in \omega$. It is clear that $\mathfrak{E}_{\sigma\restriction k}$ converges to the point $\sigma \in \mathbb{Q}^\omega$ in the product topology of $\mathbb{Q}^\omega$, hence it suffices to show that $\sigma \in \mathfrak{E}$. Since B is bounded there is an $M \in \mathbb{N}$ such that $B \subset \{x \in \mathbb{Q}^\omega : \|x\| \leq M\}$ and because $\mathfrak{E}_{\sigma\restriction k} \cap B \neq \emptyset$ for all $k \in \omega$ this means that $\|(q_0, q_1, \ldots, q_{k-1}, 0, 0, \ldots)\| \leq M$ for all $k \geq 0$. Since the norm function is LSC on $\mathbb{Q}^\omega$ we have

$$\|\sigma\| = \lim_{k\to\infty} \|(q_0, q_1, \ldots, q_{k-1}, 0, 0, \ldots)\| \leq M,$$

so $\sigma \in \mathfrak{E}$.

Dijkstra and van Mill [11, Theorem 8.13] proved the following characterization of $\mathfrak{E}$.

Theorem 1.3.5. *A nonempty space E is homeomorphic to $\mathfrak{E}$ if and only if there exists a topology $\mathfrak{T}$ on E that witnesses the almost zero-dimensionality of E and there exist a nonempty tree T over a countable set and subspaces E_s of E that are closed with respect to $\mathfrak{T}$ for each $s \in T$ such that:*

(1) $E_\lambda = E$ *and* $E_s = \bigcup\{E_t : t \in \mathrm{succ}(s)\}$ *whenever* $s \in T$*;*

(2) *each* $x \in E$ *has a neighbourhood* U *that is an anchor for* $(E_s)_{s\in T}$ *in* $(E, \mathcal{T})$*;*

(3) *for each* $s \in T$ *and* $t \in \mathrm{succ}(s)$ *the set* E_t *is nowhere dense in* E_s*; and*

(4) E *is* $\{E_s : s \in T\}$*-cohesive.*

We show that $\mathfrak{E}$ satisfies the conditions of Theorem 1.3.5. Let the topology $\mathcal{T}$ on $\mathfrak{E}$, the tree T, and the subspaces $\mathfrak{E}_s \subset \mathfrak{E}$ for $s \in T$ be as in Example 1.3.4. It is clear that $\mathfrak{E}_s$ is closed in $(\mathfrak{E}, \mathcal{T})$ for all $s \in T$ and conditions (1) and (3) are easily seen to be satisfied. Furthermore, it follows from Example 1.3.4 that every bounded neighbourhood of a point $x \in \mathfrak{E}$ is an anchor for $(\mathfrak{E}_s)_{s\in T}$ in $(\mathfrak{E}, \mathcal{T})$, so condition (2) is satisfied. Finally, as noted before, it follows from Corollary 1.2.6 that every nonempty clopen subset of $\mathfrak{E}_s$ is unbounded. This means that a bounded neighbourhood of a point $x \in \mathfrak{E}$ also does not contain nonempty clopen subsets of any space $\mathfrak{E}_s$, hence condition (4) is satisfied as well.

A topology $\mathcal{T}$ on a space Z is called an *$F_{\sigma\delta}$-topology* on Z if $(Z, \mathcal{T})$ is an (absolute) $F_{\sigma\delta}$-space. In [11, Theorem 8.13] Dijkstra and van Mill also give the following topological characterization of $\mathfrak{E}$.

Theorem 1.3.6. *A nonempty space* E *is homeomorphic to* $\mathfrak{E}$ *if and only if there exists an* $F_{\sigma\delta}$*-topology* $\mathcal{T}$ *on* E *that witnesses the almost zero-dimensionality of* E *and there exist a nonempty tree* T *over a countable set and subspaces* E_s *of* E *that are closed with respect to* $\mathcal{T}$ *for each* $s \in T \setminus \{\lambda\}$ *such that*

(1′) E_λ *is dense in* E *and* $E_s = \bigcup\{E_t : t \in \mathrm{succ}(s)\}$ *whenever* $s \in T$*;*

(2′) *each* $x \in E$ *has a neighbourhood* U *that is an anchor for* $(E_s)_{s\in T}$ *in* $(E, \mathcal{T})$*;*

(3′) *for each* $s \in T \setminus \lambda$ *and* $t \in \mathrm{succ}(s)$ *the set* E_t *is nowhere dense in* E_s*;*

(4′) E *is* $\{E_s : s \in T\}$*-cohesive; and*

(5′) *E can be written as a countable union of nowhere dense subsets that are closed with respect to* $\mathcal{T}$.

As an illustration we show that $\mathfrak{E}$ satisfies the conditions of Theorem 1.3.6. Again, let $\mathcal{T}$ be the product topology that $\mathfrak{E}$ inherits from $\mathbb{Q}^\omega$, put $T = \mathbb{Q}^{<\omega}$ and let $\mathfrak{E}_s$ for $s \in T$ be as defined in Example 1.3.4. We already showed after Theorem 1.3.5 that with these choices for the topology $\mathcal{T}$ on $\mathfrak{E}$, the tree T and the subspaces $\mathfrak{E}_s \subset \mathfrak{E}$, the conditions of Theorem 1.3.5 are satisfied. This immediately implies that $\mathfrak{E}$ satisfies conditions (1′)–(5′) of Theorem 1.3.6. Furthermore, since $\mathbb{Q}$ is a σ-compact space, it is easy to see that $\mathbb{Q}^\omega$ is an absolute $F_{\sigma\delta}$-space. Noting that $\mathfrak{E}$ is an F_σ subset of $\mathbb{Q}^\omega$, we see that $\mathcal{T}$ is indeed an $F_{\sigma\delta}$-topology on $\mathfrak{E}$. This shows that $\mathfrak{E}$ satisfies Theorem 1.3.6.

At first glance there appears to be not much difference between Theorem 1.3.5 and Theorem 1.3.6. However, this is an false impression. When we want to apply Theorem 1.3.5 to prove that a space E is homeomorphic to $\mathfrak{E}$, condition (1) requires us to construct a stratification of the entire space, whereas condition (1′) of Theorem 1.3.6 requires only a stratification of a dense subset of E. This can make life considerably easier. Consider for example the case that the space E, which is possibly homeomorphic to $\mathfrak{E}$, is a topological group. Then we need only three things to satisfy Theorem 1.3.6: an $F_{\sigma\delta}$ witness topology $\mathcal{T}$ with the property that group translations are homeomorphisms with respect to $\mathcal{T}$, the first category property (5′), and a suitable closed imbedding of $\mathfrak{E}$ in E. For if we have a copy $\mathcal{E}$ of $\mathfrak{E}$ in E of the right type which means in particular that the imbedding is also closed relative to the respective witness topologies, then we can obtain the dense stratified set E_λ by simply multiplying $\mathcal{E}$ with a countable dense subset of the group E. This strategy is used by Dijkstra and van Mill [11, Chapter 10] to classify homeomorphism groups and here we will use it to prove Theorem 2.4.1. This theorem states that for $n \in \mathbb{N}\setminus\{3\}$ certain homeomorphism groups of M_n^{n+1}, the n-dimensional Menger continuum in $\mathbb{R}^{n+1}$ (see Engelking [18, §1.11]), also known as the n-dimensional Sierpiński carpet, are homeomorphic to $\mathfrak{E}$. The proof of this theorem is based on the proof of [11, Theorem 10.4] where Dijkstra and van Mill derive a similar result for the universal Menger continuum of dimension $m \in \mathbb{N}$ (see Engelking [18, §1.11]). The suitable imbeddings of

Erdős space in the homeomorphism groups mentioned in Theorem 2.4.1 come from the imbedding of a copy of $\mathfrak{E}_{\mathrm{c}}$ in the space $\mathcal{H}(M_n^{n+1})$, for $n \in \mathbb{N} \setminus \{3\}$, constructed by Dijkstra [7, Theorem 5]. In this article Dijkstra uses this imbedding to show that $\dim \mathcal{H}(M_n^{n+1}) = 1$ if $n \in \mathbb{N} \setminus \{3\}$.

Chapter 2

Homeomorphism groups of Sierpiński carpets and Erdős space

2.1 Introduction

In this chapter we will study certain homeomorphism groups of Sierpiński carpets. Remember that for a locally compact space X we equip the group of autohomeomorphisms $\mathcal{H}(X)$ of X with the compact-open topology. For a subset A of X we let $\mathcal{H}(X, A)$ denote the subgroup $\{h \in \mathcal{H}(X) : h(A) = A\}$ of $\mathcal{H}(X)$.

Let D be a countable dense subset of a locally compact space X. Dijkstra and van Mill [11] showed that if X contains a nonempty open subset homeomorphic to $\mathbb{R}^n$ for $n \geq 2$, an open subset of the Hilbert cube , or an open subset of some universal Menger continuum μ^n for $n \in \mathbb{N}$, then $\mathcal{H}(X, D)$ is homeomorphic to $\mathfrak{E}$. In line with these results we consider the topological group $\mathcal{H}(M_n^{n+1}, D)$ for $n \in \mathbb{N}$. Here M_n^{n+1} is the n-dimensional Menger continuum in $\mathbb{R}^{n+1}$, also known as the n-dimensional Sierpiński carpet, and D is a countable dense subset of M_n^{n+1}. In our main result of this chapter, Theorem 2.4.1, we show that under some appropriate conditions on D we have that $\mathcal{H}(M_n^{n+1}, D)$ is homeomorphic to $\mathfrak{E}$ for $n \in \mathbb{N} \setminus \{3\}$. The proof of this result is based on the proof of [11, Theorem 10.4] where Dijkstra and van Mill use one of their their characterizations of $\mathfrak{E}$ to deal with the μ^n case. We also heavily rely on Dijkstra [7, §5] where it is shown that there are closed imbeddings of $\mathfrak{E}_c$ in the space $\mathcal{H}(M_n^{n+1})$, if $n \in \mathbb{N} \setminus \{3\}$. The main complication is that M_n^{n+1} is, in contrast to the n-dimensional universal Menger continuum considered by Dijkstra and van Mill in [11, Theorem 10.4], not homogeneous. *Unless stated otherwise every space is separable and metrizable in this chapter.*

2.2 Preliminaries

Let $\mathbb{R}^+ = [0, \infty)$. We shall use a number of compactifications of $\mathbb{R}^m$. Let S^m denote the one-point compactification of $\mathbb{R}^m$. Remember that $\hat{\mathbb{R}}$ denotes the compactification $[-\infty, \infty]$ of $\mathbb{R}$. We use the convention that $\pm\infty + t = \pm\infty$ when $t \in \mathbb{R}$. This extends the addition operation on $\mathbb{R}^m$ to a continuous function from $\hat{\mathbb{R}}^m \times \mathbb{R}^m$ to $\hat{\mathbb{R}}^m$. Let $\pi_i \colon \hat{\mathbb{R}}^m \to \hat{\mathbb{R}}$ for $i \in \omega$ be the coordinate projection given by $\pi_i(x) = x_i$. An *m-cell* is any space that is homeomorphic to I^m, where $I = [0, 1]$. Finally, if $m \geq 2$, we let Q^m stand for the quotient space obtained from $\hat{\mathbb{R}}^m$ by identifying the faces $\{x \in \hat{\mathbb{R}}^m : x_2 = \infty\}$ and $\{x \in \hat{\mathbb{R}}^m : x_2 = -\infty\}$ to the points α and β, respectively. Note that Q^m is like $\mathbb{R}^m$ an m-cell that compactifies $\mathbb{R}^m$.

Recall that for a compact space X the compact-open topology on $\mathcal{H}(X)$ coincides with the topology of uniform convergence. We write e_X for the identity element of $\mathcal{H}(X)$. If O is an open subset of X then we say that $h \in \mathcal{H}(X)$ is *supported on O* if h is equal to the identity on $X \setminus O$, i.e. if $h\restriction(X \setminus O) = e_{X\setminus O}$. We write $\mathcal{H}_O(X)$ for the subgroup of $\mathcal{H}(X)$ consisting of all homeomorphisms of X that are supported on O, so $\mathcal{H}_O(X) = \{h \in \mathcal{H}(X) : h\restriction(X \setminus O) = e_{X\setminus O}\}$. Furthermore, we let $\mathcal{H}_O(X, A)$ denote the subgroup $\mathcal{H}_O(X) \cap \mathcal{H}(X, A)$ of $\mathcal{H}(X)$.

We will now recall the construction of the *n-dimensional Menger continuum M_n^m in* $\mathbb{R}^m$ for $n, m \in \mathbb{N}$ with $n < m$. Start with the cube $C_0 = I^m$ and divide it into 3^m congruent subcubes. The n-skeleton of C_0 is the union of all faces of C_0 which have dimension at most n. The space C_1 is the subspace of C_0 obtained by taking the union of all the subcubes that meet the n-skeleton of C_0. Repeat this process on each of the cubes that make up C_1 and so on. In this way we obtain a sequence $C_0 \supset C_1 \supset C_2 \ldots$ and we put $M_n^m = \bigcap_{i=0}^{\infty} C_i$. See Engelking [18, §1.11] for details.

Let X be a compact metrizable space. A sequence $(A_n)_{n\in\omega}$ of subsets of X is called a *null sequence* if $\lim_{n\to\infty} \operatorname{diam} A_n = 0$. Note that this is a valid definition, that is, it does not depend on the chosen metric for X, because of the well known result that any two compatible metrics on a compact space are uniformly equivalent.

We give the definition of an n-dimensional Sierpiński carpet.

Definition 2.2.1. Let $n \in \mathbb{N}$. A nowhere dense subset X of S^{n+1} is called an *n-dimensional Sierpiński carpet* if the collection of components $\{U_i : i \in \mathbb{N}\}$ of $S^{n+1} \setminus X$ forms a null sequence such that the closures of the U_i's are a pairwise disjoint collection and every $S^{n+1} \setminus U_i$ is an $(n+1)$-cell.

The Menger continuum M_n^{n+1} is a standard example of an n-dimensional Sierpiński carpet. The following characterization theorem is due to Whyburn [30] (for $n = 1$) and Cannon [4] (for $n \geq 2$).

Theorem 2.2.2. *Let X and Y be two n-dimensional Sierpiński carpets for $n \in \mathbb{N} \setminus \{3\}$ and let U and V be components of $S^{n+1} \setminus X$ and $S^{n+1} \setminus Y$, respectively. If h is a homeomorphism from the boundary of U to the boundary of V, then h can be extended to a homeomorphism from X to Y.*

Remark 2.2.3. In Theorem 2.2.2, let S and T be components of $S^{n+1} \setminus X$ and $S^{n+1} \setminus Y$, respectively, such that $S \neq U$ and $T \neq V$. The proofs of Lemma 1 and Theorem 1 in Cannon [4] together with the Annulus Theorem ([4]), which enables one to control where the boundary of a component of $S^{n+1} \setminus X$ is mapped to, yield that we can extend h to a homeomorphism $\overline{h}\colon X \to Y$ in such a way that $\overline{h}(\partial S) = \partial T$.

Definition 2.2.4. A point x of an n-dimensional Sierpiński carpet X is called a *boundary point* of X if it lies on a non-separating copy S of S^n in X, that is, $X \setminus S$ is connected. If x is not a boundary point we call it an *interior point* of X.

Using the notation of Definition 2.2.1, it follows easily from the generalized Jordan curve theorem, see Munkres [25, Theorem 36.3], that x is a boundary point of X if and only if $x \in \bigcup_{i=1}^{\infty} \partial U_i$. Note that these definitions of boundary point and interior point of X do not coincide with the usual meaning of these notions since $\mathrm{Int}(X) = \emptyset$. We have that boundary points and interior points are two topologically different types of points in X, both of which are represented in X. This means that X is not homogeneous. It is well known that these points are topologically the only two different types of points in X if $\dim X \neq 3$, cf. Theorem 2.2.2 and Lemma 2.2.6.

Lemma 2.2.5. *Let $n \in \mathbb{N} \setminus \{3\}$ and suppose that $x \in \partial U$, where U is a component of $S^{n+1} \setminus M_n^{n+1}$. Then there is a local basis $\mathcal{B}_x$ at x such that for every $B \in \mathcal{B}_x$ and every $y \in B \cap \partial U$ there is a homeomorphism h of M_n^{n+1} with $h(x) = y$ that is supported on B.*

Proof. Take $n \in \mathbb{N} \setminus \{3\}$ and note that it follows from Theorem 2.2.2 and the homogeneity of S^n that all boundary points of M_n^{n+1} are topologically equivalent. Therefore, it is enough to consider the boundary point $x = (0, 0, \ldots, 0) \in \partial(I^{n+1})$, where $\partial(I^{n+1})$ is the boundary of the unbounded component of $\mathbb{R}^{n+1} \setminus M_n^{n+1}$. For $\mathcal{B}_x$ we take the collection $\{B_i : i \in \omega\}$, where $B_i = M_n^{n+1} \cap [0, 3^{-i})^{n+1}$. Now take $i \in \omega$ and a point $y \in B_i \cap \partial I^{n+1}$. If $y = x$ then the identity map on M_n^{n+1} is obviously a homeomorphism that satisfies the requirements of the lemma, so we suppose that $y \neq x$. The closure of B_i in M_n^{n+1} is the space $C_i = M_n^{n+1} \cap [0, 3^{-i}]^{n+1}$, so $C_i = 3^{-i} M_n^{n+1}$, which means that C_i is again an n-dimensional Sierpiński carpet. Note that $D_i = \partial\big([0, 3^{-i}]^{n+1}\big)$ is the boundary of the unbounded component of $\mathbb{R}^{n+1} \setminus C_i$. Since $B_i \cap D_i$ is open and connected in D_i and D_i is homeomorphic to S^n, the space $B_i \cap D_i$ is path connected and we can use the strong local homogeneity of S^n to see that there is a homeomorphism $g_i \colon D_i \to D_i$ with $g_i(x) = y$ and that is supported on $B_i \cap D_i$. By Theorem 2.2.2 we can extend g_i to a homeomorphism $\overline{g_i}$ of C_i. If we now define $h_i \colon M_n^{n+1} \to M_n^{n+1}$ by

$$h_i(x) = \begin{cases} \overline{g_i}(x), & \text{if } x \in C_i; \\ x, & \text{if } x \notin C_i, \end{cases}$$

then h_i is as required. $\square$

We want to derive a similar result for the interior points of M_n^{n+1} with $n \in \mathbb{N} \setminus \{3\}$. For this we use the following lemma. Remember that $\partial(I^{n+1})$ is the boundary of the unbounded component of $\mathbb{R}^{n+1} \setminus M_n^{n+1}$.

Lemma 2.2.6. *Let $n \in \mathbb{N} \setminus \{3\}$ and let x and y be interior points of M_n^{n+1}. Then there is a homeomorphism $h \colon M_n^{n+1} \to M_n^{n+1}$ with $h(x) = y$ and $h \restriction \partial(I^{n+1}) = e_{\partial(I^{n+1})}$.*

Proof. If $x = y$ we can take $h = e_{M_n^{n+1}}$, so suppose that $x \neq y$. Clearly, we can find quotient mappings $q_x, q_y \colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with $q_x^{-1}(\{x\}) = q_y^{-1}(\{y\}) = I^{n+1}$ and such that $q_x \colon \mathbb{R}^{n+1} \setminus I^{n+1} \to \mathbb{R}^{n+1} \setminus \{x\}$ and $q_y \colon \mathbb{R}^{n+1} \setminus I^{n+1} \to \mathbb{R}^{n+1} \setminus \{y\}$ are homeomorphisms. Then $q_x^{-1}(M_n^{n+1}) \setminus \operatorname{Int} I^{n+1}$ and $q_y^{-1}(M_n^{n+1}) \setminus \operatorname{Int} I^{n+1}$ are Sierpiński carpets and we denote them by S_x and S_y, respectively.

Let B_x, respectively B_y, be the boundary of the unbounded component of $\mathbb{R}^{n+1} \setminus S_x$, respectively $\mathbb{R}^{n+1} \setminus S_y$. So $B_x = q_x^{-1}(\partial I^{n+1})$ and $B_y = q_y^{-1}(\partial I^{n+1})$. Note that $g = (q_y^{-1} \circ q_x) \restriction B_x$ is a homeomorphism from B_x to B_y such that $q_y \circ g = q_x \restriction B_x$. It follows from Remark 2.2.3 that we can extend g to a homeomorphism $\overline{g} \colon S_x \to S_y$ such that $g(\partial I^{n+1}) = \partial I^{n+1}$.

Now define the function $h \colon M_n^{n+1} \to M_n^{n+1}$ by

$$h(z) = \begin{cases} y, & \text{if } z = x; \\ \big(q_y \circ \overline{g} \circ q_x^{-1}\big)(z), & \text{if } z \neq x. \end{cases}$$

It is easy to see that h is a bijection such that $h \circ q_x = q_y \circ \overline{g}$. Since q_x is a quotient mapping and $q_y \circ \overline{g}$ is continuous the function h is continuous. By compactness of M_n^{n+1} we see that h is a homeomorphism.

Take $z \in \partial(I^{n+1})$. Then $q_x^{-1}(z) \in B_x$ and since $\overline{g}$ is an extension of g we see that

$$h(z) = (q_y \circ \overline{g})(q_x^{-1}(z)) = (q_y \circ g)(q_x^{-1}(z)) = q_x(q_x^{-1}(z)) = z.$$

This shows that $h \restriction \partial(I^{n+1}) = e_{\partial(I^{n+1})}$, so h is as required. $\square$

Lemma 2.2.7. *Let $n \in \mathbb{N} \setminus \{3\}$ and suppose that x is an interior point of M_n^{n+1}. Then there is a local basis $\mathcal{B}_x$ at x such that for every $B \in \mathcal{B}_x$ and every interior point y of M_n^{n+1} in B there is a homeomorphism h of M_n^{n+1} with $h(x) = y$ that is supported on B.*

Proof. Let x be an interior point of M_n^{n+1}. It follows from the construction of M_n^{n+1} that x has arbitrarily small open neighbourhoods B in M_n^{n+1} such that $\overline{B}$, the closure of B in M_n^{n+1} (or in $\mathbb{R}^{n+1}$), is homeomorphic to M_n^{n+1} and the boundary ∂B of B in M_n^{n+1} is the boundary of the unbounded component of $\mathbb{R}^{n+1} \setminus \overline{B}$. Let $\mathcal{B}_x$ be the collection of these neighbourhoods B of x. Clearly, $\mathcal{B}_x$ is a local basis at x. If y is an interior point of M_n^{n+1} such that y is an element of a set $B \in \mathcal{B}_x$, then y is an interior point of $\overline{B}$. It follows from Lemma 2.2.6 that we can find a homeomorphism of $\overline{B}$ that maps x onto y and that is equal to the identity on the boundary of B in M_n^{n+1}. This homeomorphism can be extended to M_n^{n+1} by taking the identity on $M_n^{n+1} \setminus \overline{B}$. We showed that the local basis $\mathcal{B}_x$ at x is as required. □

Lemma 2.2.8. *Let O be an open subset of M_n^{n+1} for $n \in \mathbb{N} \setminus \{3\}$ and let D_1 and D_2 be countable subsets of O. Suppose that for $j \in \{1, 2\}$ the interior points of M_n^{n+1} contained in D_j are dense in O and $D_j \cap \partial U_i$ is dense in $\partial U_i \cap O$ for all i. Then there is a homeomorphism h of M_n^{n+1} that is supported on O and that satisfies $h(D_1) = D_2$.*

Proof. This proof is similar to the proof of van Mill [24, Theorem 1.6.9]. Write $D_1 = D_1^{\mathrm{i}} \cup D_1^{\mathrm{b}}$, where D_1^{i} is the set of all points of D_1 that are interior points of M_n^{n+1} and D_1^{b} is the set of all points of D_1 that are boundary points of M_n^{n+1}. Similarly, write $D_2 = D_2^{\mathrm{i}} \cup D_2^{\mathrm{b}}$. Let $\{a_1, a_2, \ldots\}$ and $\{\tilde{a}_1, \tilde{a}_2, \ldots\}$ be enumerations of D_1^{i} and D_1^{b}, respectively, and let $\{b_1, b_2, \ldots\}$ and $\{\tilde{b}_1, \tilde{b}_2, \ldots\}$ be enumerations of D_2^{i} and D_2^{b}, respectively. Using the Inductive Convergence Criterion (see van Mill [24, 1.6.2]) we construct a sequence $(h_m)_{m \in \mathbb{N}}$ of homeomorphisms of M_n^{n+1} such that $h = \lim_{m \to \infty} h_m \circ \cdots \circ h_1$ exists and is a homeomorphism and such that the following conditions are satisfied:

(1) h_m is supported on O for all $m \in \mathbb{N}$;

(2) $h_m \circ \cdots \circ h_1(a_j) = h_{4j-2} \circ \cdots \circ h_1(a_j) \in D_2^{\mathrm{i}}$ for all j and $m \geq 4j - 2$;

(3) $(h_m \circ \cdots \circ h_1)^{-1}(b_j) = (h_{4j-1} \circ \cdots \circ h_1)^{-1}(b_j) \in D_1^{\mathrm{i}}$ for all j and $m \geq 4j - 1$;

(4) $h_m \circ \cdots \circ h_1(\tilde{a}_j) = h_{4j} \circ \cdots \circ h_1(\tilde{a}_j) \in D_2^{\mathrm{b}}$ for all j and $m \geq 4j$;

(5) $(h_m \circ \cdots \circ h_1)^{-1}(\tilde{b}_j) = (h_{4j+1} \circ \cdots \circ h_1)^{-1}(\tilde{b}_j) \in D_1^{\mathrm{b}}$ for all j and $m \geq 4j + 1$.

These conditions ensure that $h \in \mathcal{H}_O(M_n^{n+1})$ and that $h(D_1^{\mathrm{i}}) = D_2^{\mathrm{i}}$ and $h(D_1^{\mathrm{b}}) = D_2^{\mathrm{b}}$. Put $h_1 = e_{M_n^{n+1}}$ and assume that $h_1, \ldots, h_{4j-3}$ are defined for certain $j \in \mathbb{N}$.

If $h_{4j-3} \circ \cdots \circ h_1(a_j) \in D_2^{\mathrm{i}}$, take $h_{4j-2} = e_{M_n^{n+1}}$. Otherwise, we use Lemma 2.2.7 to find a small neighbourhood $V_{4j-2} \subset O$ of $h_{4j-3} \circ \cdots \circ h_1(a_j)$ which is disjoint from the finite set

$$\{b_1, \ldots, b_{j-1}, \tilde{b}_1, \ldots, \tilde{b}_{j-1}\} \cup h_{4j-3} \circ \cdots \circ h_1(\{a_1, \ldots, a_{j-1}, \tilde{a}_1, \ldots \tilde{a}_{j-1}\})$$

and moreover has the property that we can map $h_{4j-3} \circ \cdots \circ h_1(a_j)$ to every other interior point of M_n^{n+1} in V_{4j-2} with a homeomorphism that is supported on V_{4j-2}. Since D_2^{i} is dense in O we have $D_2^{\mathrm{i}} \cap V_{4j-2} \neq \emptyset$. This means that we can find a homeomorphism f_{4j-2} of M_n^{n+1} supported on V_{4j-2} such that

$$f_{4j-2} \circ h_{4j-3} \circ \cdots \circ h_1(a_j) \in D_2^{\mathrm{i}}.$$

We put $h_{4j-2} = f_{4j-2}$.

If $(h_{4j-2} \circ \cdots \circ h_1)^{-1}(b_j) \in D_1^{\mathrm{i}}$, we take $h_{4j-1} = e_{M_n^{n+1}}$. Otherwise, we use Lemma 2.2.7 again to find a small neighbourhood $V_{4j-1} \subset O$ of b_j that is disjoint from the finite set

$$\{b_1, \ldots, b_{j-1}, \tilde{b}_1, \ldots, \tilde{b}_{j-1}\} \cup h_{4j-2} \circ \cdots \circ h_1(\{a_1, \ldots, a_j, \tilde{a}_1, \ldots \tilde{a}_{j-1}\})$$

and has the property that we can map b_j on every other interior point of M_n^{n+1} in V_{4j-1} with a homeomorphism that is supported on V_{4j-1}. Since $h_{4j-2} \circ \cdots \circ h_1(D_1^{\mathrm{i}})$ is dense in O by property (1) we know that $h_{4j-2} \circ \cdots \circ h_1(D_1^{\mathrm{i}}) \cap V_{4j-1} \neq \emptyset$. This means that there is a homeomorphism f_{4j-1} of M_n^{n+1} supported on V_{4j-1} such that

$$f_{4j-1}^{-1}(b_j) \in h_{4j-2} \circ \cdots \circ h_1(D_1^{\mathrm{i}}).$$

We put $h_{4j-1} = f_{4j-1}$.

Using the same argumentation as above, now using Lemma 2.2.5 instead of Lemma 2.2.7, we find appropriate neighbourhoods $V_{4j} \subset O$ and $V_{4j+1} \subset O$ of the points $h_{4j-1} \circ \cdots \circ h_1(\tilde{a}_j)$ and $\tilde{b}_j$, respectively, and homeomorphisms $h_{4j} \in \mathcal{H}_{V_{4j}}(M_n^{n+1})$ and $h_{4j+1} \in \mathcal{H}_{V_{4j+1}}(M_n^{n+1})$, such that

$$h_{4j} \circ h_{4j-1} \circ \cdots \circ h_1(\tilde{a}_j) \in D_2^{\mathrm{b}},$$

and

$$h_{4j+1}^{-1}(\tilde{b}_j) \in h_{4j} \circ \cdots \circ h_1(D_1^{\mathrm{b}}).$$

If the neighbourhoods $V_{4j-2}, V_{4j-1}, V_{4j}$, and V_{4j+1} are chosen small enough, the conditions of the Inductive Convergence Criterion are satisfied. $\square$

Remark 2.2.9. It follows immediately from this lemma that if $D_1 \cap \partial U_i$ and $D_2 \cap \partial U_i$ are dense in $\partial U_i \cap O$ for every i and D_1 and D_2 do not contain any interior points of M_n^{n+1}, there is a homeomorphism h of M_n^{n+1} that is supported on O that maps D_1 onto D_2. Similarly, if D_1 and D_2 both consist entirely of interior points of M_n^{n+1} there is a homeomorphism that maps D_1 onto D_2 which is supported on O.

We repeat Theorem 1.3.6 taken from Dijkstra and van Mill [11].

Theorem 2.2.10. *A nonempty space E is homeomorphic to $\mathfrak{E}$ if and only if there exists an $F_{\sigma\delta}$-topology $\mathcal{T}$ on E that witnesses the almost zero-dimensionality of E and there exist a nonempty tree T over a countable set and subspaces E_s of E that are closed with respect to $\mathcal{T}$ for each $s \in T \setminus \{\lambda\}$ such that*

(1′) *E_λ is dense in E and $E_s = \bigcup\{E_t : t \in \mathrm{succ}(s)\}$ whenever $s \in T$;*

(2′) *each $x \in E$ has a neighbourhood U that is an anchor for $(E_s)_{s\in T}$ in $\big(E, \mathcal{T}\big)$;*

(3′) *for each $s \in T \setminus \lambda$ and $t \in \mathrm{succ}(s)$ the set E_t is nowhere dense in E_s;*

(4′) *E is $\{E_s : s \in T\}$-cohesive; and*

(5′) *E can be written as a countable union of nowhere dense subsets that are closed with respect to $\mathcal{T}$.*

In §1.3 we showed that $\mathfrak{E}$ satisfies the conditions of Theorem 2.2.10.

2.3 Imbedding complete Erdős space

This section is based on Dijkstra [7, §5]. There it is explained how to construct an imbedding of a particular copy of $\mathfrak{E}_c$ in $\mathcal{H}(M_n^{n+1})$ if $n \in \mathbb{N} \setminus \{3\}$. This shows that $\dim \mathcal{H}(M_n^{n+1}) = 1$ for $n \in \mathbb{N} \setminus \{3\}$

because Oversteegen and Tymchatyn [26] already proved that $\dim \mathcal{H}(M_n^{n+1}) \leq 1$ for all $n \in \mathbb{N}$. The reason that the case $n = 3$ is left out is because Theorem 2.2.2 is used in the construction of the imbedding. Our aim here is to show the construction of this imbedding and to prove some properties of it that will be of importance for Theorem 2.4.1, our main result. However, for details one should consult [7, §5]. We write $\mathbf{0}$ for the zero vector in $\mathbb{R}^{n+1}$ for $n \in \mathbb{N}$ and we use the same notation for the zero vector in $\mathbb{R}^\omega$.

Consider the space

$$E_2 = \{z \in \ell^1 : 2^i z_i \in \omega \text{ for all } i \in \omega\}. \tag{2.1}$$

It follows from Corollary 1.2.6 and Theorem 1.2.7 that E_2 is homeomorphic to $\mathfrak{E}_{\mathrm{c}}$. Remember that Q^m for $m \geq 2$ is the quotient space obtained from $\hat{\mathbb{R}}^m$ by identifying the faces $\{x \in \hat{\mathbb{R}}^m : x_2 = \infty\}$ and $\{x \in \hat{\mathbb{R}}^m : x_2 = -\infty\}$ to the points α and β, respectively. Take $n \in \mathbb{N} \setminus \{3\}$ and let $u = (1, 0, \ldots)$ and $v = (0, 1, \ldots)$ be unit vectors in $\mathbb{R}^{n+1}$. We will construct a topological copy $\overline{B}$ of M_n^{n+1} in Q^{n+1} that contains the set $\{\mathbf{0}\} \cup \{2^{-i+1}u : i \in \omega\}$. Then we will construct an imbedding $\overline{H} \colon E_2 \to \mathcal{H}(\overline{B})$ such that for each $z \in E_2$ we have:

$$\overline{H}_z(\mathbf{0}) = \|z\|_1 v \quad \text{and} \quad \overline{H}_z(2^{-i+1}u) = 2^{-i+1}u + \sum_{i=0}^{i-1} z_k v$$

for every $i \in \omega$. So on the points of the set $\{\mathbf{0}\} \cup \{2^{-i+1}u : i \in \omega\}$ the map $\overline{H}$ is a vertical shift. We will now concentrate on constructing a 'shear transformation' σ that we will use to connect these vertical shifts with each other.

Let $C = [1, 2] \times I^n$ be a unit cube in $\mathbb{R}^{n+1}$. We split C into two triangular cells:

$$T_1 = \{x \in C : x_1 + x_2 \leq 2\} \quad \text{and} \quad T_2 = \{x \in C : x_1 + x_2 \geq 2\}.$$

Note that $D = T_2 \cup T_1 + v$ is also an $(n+1)$-cell and $\varphi \colon C \to D$ defined by

$$\varphi(x) = x + (2 - x_1)v$$

is a homeomorphism. Let S_1 and S_2 be n-dimensional Sierpiński carpets that are obtained by deleting a suitable null sequence of open sets U_i

from the interiors of T_1 and T_2, respectively. Note that $K = S_1 \cup S_2$ and $L = S_2 \cup (S_1 + v)$ are also n-dimensional Sierpiński carpets such that ∂C and ∂D are the boundaries of the unbounded components of $\mathbb{R}^{n+1} \setminus K$ and $\mathbb{R}^{n+1} \setminus L$, respectively. According to Theorem 2.2.2 there is a homeomorphism $\sigma\colon K \to L$ that extends $\varphi \restriction \partial C$. We define closed subsets N and F of $\mathbb{R}^{n+1}$ as follows:

$$N = C + \{kv : k \in \mathbb{Z}\} = D + \{kv : k \in \mathbb{Z}\} = [1,2] \times \mathbb{R} \times I^{n+1}$$

and

$$F = K + \{kv : k \in \mathbb{Z}\} = L + \{kv : k \in \mathbb{Z}\} \subset N.$$

We can extend σ to an autohomeomorphism of F by the rule

$$\sigma(x + kv) = \sigma(x) + kv$$

for $x \in K$ and $k \in \mathbb{Z}$. Note that if $x_1 = 2$, then $\sigma(x) = x$ and if $x_1 = 1$, then $\sigma(x) = x + v$. Thus the map σ is a shear transformation that connects the identity on the 'right face' of F with a vertical shift by 1 unit on the 'left' face of F. In particular $\sigma(2u) = 2u$ and $\sigma(u) = u + v$.

Let $i \in \omega$. We introduce reduced copies of N and F:

$$N_i = 2^{-i}N \quad \text{and} \quad F_i = 2^{-i}F. \tag{2.2}$$

We also define $\sigma_i\colon F_i \to F_i$ by

$$\sigma_i(2^{-i}x) = 2^{-i}\sigma(x) \quad \text{for } x \in F.$$

Note that $\pi_1(N_i) = \pi_1(F_i) = [2^{-i}, 2^{-i+1}]$. For elements of the 'left and right faces' of F_i we have: $\sigma_i(x) = x$ whenever $x_1 = 2^{-i+1}$, and $\sigma_i(x) = x + 2^{-i}v$ whenever $x_1 = 2^{-i}$. Furthermore, it is easily seen that $F_i + 2^{-i}kv = F_i$ for every $k \in \mathbb{Z}$.

For $i \in \omega$ and $z \in E_2$ we define the following autohomeomorphism H_z^i of F_i: if $x \in F_i$, then

$$H_z^i(x) = (\sigma_i)^j(x) + \sum_{k=0}^{i-1} z_k v, \tag{2.3}$$

where $j = 2^i z_i$. We note here that $j \in \omega$ and $\sum_{k=0}^{i-1} z_k \in 2^{-i}\mathbb{Z}$ by definition of E_2, which ensures that $H_z^i(x) \in F_i$. Observe that

$$H_z^i(2^{-i+1}u) = 2^{-i+1}u + \sum_{k=0}^{i-1} z_k v \tag{2.4}$$

and

$$H_z^i(2^{-i}u) = 2^{-i}u + \sum_{k=0}^{i} z_k v. \tag{2.5}$$

The shear transformation σ_i is used to connect these two translations on opposite faces of F_i.

We now take the following unions:

$$A = \bigcup_{i=0}^{\infty} N_i \quad \text{and} \quad B = \bigcup_{i=0}^{\infty} F_i.$$

Let V stand for the line $\mathbb{R}v = \{0\} \times \mathbb{R} \times \{0\} \times \cdots \times \{0\}$ in $\mathbb{R}^{n+1}$. We let $\overline{A}$ and $\overline{B}$ be the closures of A and B, respectively, in Q^{n+1}. Note that $\overline{A} = A \cup V \cup \{\alpha, \beta\}$ and $\overline{B} = B \cup V \cup \{\alpha, \beta\}$. Clearly, $A_0 = \{x \in \overline{A} : x_2 = 0\}$ is an n-cell and $\overline{A}$ is obtained from A_0 by identifying the faces $A_0 \times \{-\infty\}$ and $A_0 \times \{\infty\}$ in $A_0 \times \hat{\mathbb{R}}$ to the points α and β, respectively, in Q^{n+1}, which means that $\overline{A}$ is an $(n+1)$-cell. We can now apply Theorem 2.2.2 to see that $\overline{B}$ is homeomorphic to M_n^{n+1}.

For every $z \in E_2$ we define the autohomeomorphism H_z of B by

$$H_z = \bigcup_{i=0}^{\infty} H_z^i.$$

We extend H_z to an autohomeomorphism $\overline{H}_z$ of $\overline{B}$ by defining

$$\overline{H}_z(x) = \begin{cases} H_z(x), & \text{if } x \in B; \\ x + \|z\|_1 v, & \text{if } x \in V; \\ \alpha, & \text{if } x = \alpha; \\ \beta, & \text{if } x = \beta. \end{cases} \tag{2.6}$$

In [7, Theorem 5] Dijkstra shows that $\overline{H}\colon E_2 \to \mathcal{H}(\overline{B})$ is indeed an imbedding. Observe that since $H_z^i(x)$ depends only on the coordinates $z_0, \ldots, z_i$ of z for fixed x the image $\overline{H}_z(x)$ depends on only finitely many coordinates of $z \in E_2$. Let $p\colon Q^{n+1} \to \hat{\mathbb{R}}$ be the continuous function that is defined by

$$p(x) = \begin{cases} x_2, & \text{if } x \in \hat{\mathbb{R}} \times \mathbb{R} \times \hat{\mathbb{R}}^{n-1}; \\ \infty, & \text{if } x = \alpha; \\ -\infty, & \text{if } x = \beta. \end{cases} \tag{2.7}$$

We write Z_2 for the space consisting of the *set* E_2 equipped with the zero-dimensional topology that this set inherits from the product space $\mathbb{R}^\omega$, that is, the topology generated by the coordinate projections. We define the map $\psi\colon \mathcal{H}(\overline{B}) \to \overline{B}^C$ by the rule $\psi(h) = h{\restriction}C$, where $C = \{2^{-i+1}u : i \in \omega\} \cup \{\alpha\} \subset \overline{B}$. The following lemma is Remark 3 from [7].

Lemma 2.3.1. *The map $\psi \circ \overline{H}\colon Z_2 \to \psi(\mathcal{H}(\overline{B}))$ is a closed imbedding.*

Proof. It easily follows from the fact that $\overline{H}_z(\alpha) = \alpha$ and $\overline{H}_z(2^{-i+1}u) = 2^{-i+1}u + \sum_{k=0}^{i-1} z_k v$ for $z \in Z_2$ and $i \in \omega$ that the map $\psi \circ \overline{H}\colon Z_2 \to \psi(\mathcal{H}(\overline{B}))$ is an imbedding. Let $h \in \mathcal{H}(\overline{B})$ be such that there is a sequence $z^1, z^2, \ldots \in Z_2$ such that $\lim_{j\to\infty} \overline{H}_{z^j}{\restriction}C = h{\restriction}C$ in $\overline{B}^C$. Since $2u$ and α are elements of C this immediately implies that $h(2u) = 2u$ and $h(\alpha) = \alpha$. Combining the fact that $h(\alpha) = \alpha$ with (2.4) and (2.5) we see that for every $i \in \omega$,

$$\begin{aligned} z_i &= p(h(2^{-i}u)) - p(h(2^{-i+1}u)) \\ &= \lim_{j\to\infty} (p(\overline{H}_{z^j}(2^{-i}u)) - p(\overline{H}_{z^j}(2^{-i+1}u))) \\ &= \lim_{j\to\infty} z_i^j \end{aligned}$$

is a well defined real number. Since $z_i^j \in 2^{-i}\omega$ and this set is closed in $\mathbb{R}$, we have $z_i \in 2^{-i}\omega$ for every $i \in \omega$. Note that $h(\mathbf{0}) = \lim_{i\to\infty} h(2^{-i}u) \in [0,\infty)v$ because $h(\mathbf{0}) \neq h(\alpha)$. By definition of the z_k's we have $p(h(2^{-i+1}u)) = \sum_{k=0}^{i-1} z_k$ which means that

$$\|z\|_1 = \lim_{i\to\infty} p(h(2^{-i+1}u)) = p(h(\lim_{i\to\infty} 2^{-i+1}u)) = p(h(\mathbf{0})) < \infty. \tag{2.8}$$

Thus $z = (z_0, z_1, \ldots) \in Z_2$ and $\lim_{j\to\infty} z^j = z$ in Z_2. This implies that

$$\psi(\overline{H}_z) = \lim_{j\to\infty} \psi(\overline{H}_{z^j}) = \psi(h)$$

and the lemma is proved. □

With this result we can easily prove the following lemma.

Lemma 2.3.2. *The map $\overline{H}\colon E_2 \to \mathcal{H}(\overline{B})$ is a closed imbedding.*

PROOF. We already know that $\overline{H}\colon E_2 \to \mathcal{H}(\overline{B})$ is an imbedding, see [7, Theorem 5] for details. We now verify that it is also a closed map. Let $h \in \mathcal{H}(\overline{B})$ be such that there is a sequence $z^1, z^2, \ldots$ in E_2 with $\lim_{j\to\infty} \overline{H}_{z^j} = h$. This immediately implies that $\lim_{j\to\infty} \overline{H}_{z^j}\restriction C = h\restriction C$. With the proof of the previous lemma it follows that there is a $z \in E_2$ that is the limit of the sequence $(z^j)_j$ in Z_2 and with the property that

$$\|z\|_1 = p(h(\mathbf{0})) = \lim_{j\to\infty} p(\overline{H}_{z^j}(\mathbf{0})) = \lim_{j\to\infty} \|z^j\|_1,$$

see (2.8). Applying Proposition 1.2.1 we see that $\lim_{j\to\infty} z^j = z$ in E_2. Thus $h = \lim_{j\to\infty} \overline{H}_{z^j} = \overline{H}_z$. □

With [7, Remark 4] it follows that we have the following slight generalization of this lemma, which will be useful in the proof of Proposition 2.4.6.

Lemma 2.3.3. *Let $n \in \mathbb{N} \setminus \{3\}$ and let O be a nonempty open subset of M_n^{n+1}. Then there is a closed imbedding of E_2 in $\mathcal{H}_O(M_n^{n+1})$.*

2.4 Homeomorphism groups of a Sierpiński carpet

We prove the following theorem for n-dimensional Sierpiński carpets as an extension of the results in DIJKSTRA and VAN MILL [11, Chapter 10].

Theorem 2.4.1. *Let $n \in \mathbb{N} \setminus \{3\}$, let $\{U_i : i \in \mathbb{N}\}$ be the collection of components of $S^{n+1} \setminus M_n^{n+1}$, and let D be a countable dense subset of M_n^{n+1}. If O is a nonempty open subset of M_n^{n+1} such that either $D \cap \partial U_i = \emptyset$ for every i with $\partial U_i \subset O$ or $D \cap \partial U_i$ is dense in ∂U_i for every i with $\partial U_i \subset O$, then $\mathcal{H}_U(M_n^{n+1}, D)$ is homeomorphic to Erdős space for every open set U that contains O.*

As noted before, M_n^{n+1} is not homogeneous, which is why we need the conditions on D here. If we choose for instance a set $D \subset M_n^{n+1}$ such that $|D \cap \partial U_i| = i$ for every i then $\mathcal{H}(M_n^{n+1}, D)$ contains only the identity map.

Note that if $D \cap \partial U_i = \emptyset$ for all $\partial U_i \subset O$, there can still be a number $j \in \mathbb{N}$ with $D \cap \partial U_j \cap O \neq \emptyset$. Similarly, if $D \cap \partial U_i$ is dense in ∂U_i for all $\partial U_i \subset O$, there can still be a number $j \in \mathbb{N}$ such that $D \cap \partial U_j \cap O$ is not dense in $\partial U_j \cap O$. The following claim shows that for the proof of Theorem 2.4.1 we can avoid these situations. Furthermore, it shows that if $D \cap \partial U_i$ is dense in ∂U_i for all $\partial U_i \subset O$, we may assume that the set of interior points of M_n^{n+1} contained in $D \cap O$ is either empty or dense in O. This observation will also be useful in the proof of Theorem 2.4.1.

Claim 2.4.2. *It suffices to prove Theorem 2.4.1 for the following three cases:*

(i) $D \cap O$ *consists entirely of interior points of* M_n^{n+1}*;*

(ii) $D \cap \partial U_i \cap O$ *is dense in* $\partial U_i \cap O$ *for every* $i \in \mathbb{N}$ *and the interior points of* M_n^{n+1} *contained in* $D \cap O$ *are dense in* O*; and*

(iii) $D \cap \partial U_i \cap O$ *is dense in* $\partial U_i \cap O$ *for every* $i \in \mathbb{N}$ *and* $D \cap O$ *contains no interior points of* M_n^{n+1}*.*

PROOF. Suppose that we are in the situation of Theorem 2.4.1. Let D_{i} be the set of all points of D that are interior points of M_n^{n+1}. We define $O' \subset O$ by

$$O' = \begin{cases} O \setminus \overline{D_{\mathrm{i}}}, & \text{if } O \setminus \overline{D_{\mathrm{i}}} \neq \emptyset; \\ O, & \text{if } O \setminus \overline{D_{\mathrm{i}}} = \emptyset. \end{cases}$$

Clearly, O' is a nonempty open subset of M_n^{n+1} such that either $D_{\mathrm{i}} \cap O' = \emptyset$ or $D_{\mathrm{i}} \cap O'$ is dense in O'. Next we define $O'' \subset O'$ by

$$O'' = O' \setminus \bigcup \{\partial U_i : \partial U_i \setminus O' \neq \emptyset\}.$$

Since the interior points of M_n^{n+1} are dense in M_n^{n+1} and the collection $\{U_i : i \in \mathbb{N}\}$ forms a null sequence, O'' is a nonempty open subset of

M_n^{n+1}. Furthermore, if $\partial U_i \cap O'' \neq \emptyset$ then $\partial U_i \subset O'' \subset O$. It is clear that O'' satisfies one of the conditions (i), (ii) or (iii) and if we prove the theorem for O'' then we have also proved it for O. $\diamondsuit$

Now we focus our attention on Proposition 2.4.6, which is crucial in the proof of Theorem 2.4.1. In this proposition we work with the space E_4 rather than the space E_2 as defined in (2.1), where we define $E_4 \subset \ell^1$ by

$$E_4 = \{z \in \ell^1 : 4^i z_i \in \omega \text{ for all } i \in \omega\}. \tag{2.9}$$

In the same way as for E_2 it follows from Corollary 1.2.6 and Theorem 1.2.7 that E_4 is homeomorphic to $\mathfrak{E}_c$. Just like the definition of Z_2 given after equation (2.7), we write Z_4 for the set E_4 equipped with the zero-dimensional topology it inherits from $\mathbb{R}^\omega$. In the proof of Proposition 2.4.6 we use the imbeddings of E_2 in $\mathcal{H}(M_n^{n+1})$ for $n \in \mathbb{N} \setminus \{3\}$ that we discussed in the previous section. We first show that it easily follows from Lemma 2.3.2 that there is a closed imbedding of E_4 into $\mathcal{H}(\overline{B})$.

Imbed E_4 in E_2 by the map $g\colon E_4 \to E_2$ given by

$$g((z_0, z_1, \ldots)) = (z_0, 0, z_1, 0, z_2, 0, \ldots). \tag{2.10}$$

Note that g is even an isometry such that $g(E_4)$ is closed in Z_2 and therefore also closed in E_2. Now define the map $G\colon E_4 \to \mathcal{H}(\overline{B})$ by

$$G = \overline{H} \circ g. \tag{2.11}$$

Since g is a closed imbedding and $\overline{H}$ is a closed imbedding by Lemma 2.3.2, we see that G is a closed imbedding. We state this observation in the next lemma.

Lemma 2.4.3. *The map $G\colon E_4 \to \mathcal{H}(\overline{B})$ is a closed imbedding.*

Furthermore, it follows immediately from the fact that g is a closed imbedding that we can replace E_2 by E_4 in Lemma 2.3.3. We formulate this statement in the next lemma.

Lemma 2.4.4. *Let $n \in \mathbb{N} \setminus \{3\}$ and let O be a nonempty open subset of M_n^{n+1}. Then there is a closed imbedding of E_4 in $\mathcal{H}_O(M_n^{n+1})$.*

In Lemma 2.3.1 we showed that $\psi \circ \overline{H}\colon Z_2 \to \psi(\mathcal{H}(\overline{B}))$ is a closed imbedding. Recall that the function $\psi\colon \mathcal{H}(\overline{B}) \to \overline{B}^C$ is given by the rule $\psi(h) = h{\restriction}C$, where $C = \{2^{-i+1}u : i \in \omega\} \cup \{\alpha\} \subset \overline{B}$. Note that for $n = 1$ the points α and $2u$ are boundary points of $\overline{B}$, whereas the points $2^{-i+1}u$ for $i \geq 1$ are all interior points of $\overline{B}$. For $n \geq 2$ however, C lies in the boundary of the unbounded component of $\overline{B}$ when viewing $\overline{B}$ as a subset of $\mathbb{R}^{n+1}$. This is an easier situation than for $n = 1$. To prove Proposition 2.4.6 we need to show that for $n = 1$ there is a countable set D of boundary points of $\overline{B}$ such that the points of D converge to $\mathbf{0}$, the point α is contained in D, and such that the map

$$\psi_D \circ G\colon Z_4 \to \psi_D(\mathcal{H}(\overline{B})),$$

with $\psi_D(h) = h{\restriction}D$, is a closed imbedding. Analyzing the proof of Lemma 2.3.1 one sees that it is based on the fact that $\overline{H}_z(\alpha) = \alpha$ for all $z \in Z_2$ and the fact that $\overline{H}_z$ for $z \in Z_2$ reduces to a vertical shift on the points $2^{-i+1}u$ for $i \in \omega$, see equation (2.4). We will use this observation to prove the next lemma.

Lemma 2.4.5. *Let $n = 1$. Then there is a sequence of boundary points $(p_i)_{i\in\mathbb{N}}$ in $\overline{B}$ with $\lim_{i\to\infty} p_i = \mathbf{0}$, such that the map $\psi_D \circ G\colon Z_4 \to \psi_D(\mathcal{H}(\overline{B}))$ is a closed imbedding, where $D = \{\alpha\} \cup \{p_i : i \in \mathbb{N}\}$.*

Proof. Remember the definitions of the subsets F and F_i of $\overline{B}$ for $i \in \omega$ in §2.3. Take $z \in E_4$. By definition of g and G, see (2.10) and (2.11), we have for $x \in \overline{B}$ that

$$G_z(x) = \overline{H}_{(z_0,0,z_1,0,z_2,0,\ldots)}(x).$$

We see that it follows from (2.3) and the definition of $\overline{H}$, see (2.6), that G_z reduces to a vertical shift on F_i if i is odd:

$$G_z(x) = x + \sum_{k=0}^{(i-2)/2} z_k v, \tag{2.12}$$

for $x \in F_i$ and i an odd number.

Now take any boundary point $p \in F_1$ and put $p_1 = 2u$ and $p_i = 4^{-i+2}p$ for $i \geq 2$. Note that $p_i \in F_{2i-3}$ for every $i \geq 2$. We let D be the set $\{\alpha\} \cup \{p_i : i \in \mathbb{N}\}$. Since $G_z(\alpha) = \alpha$ and

$$G_z(p_i) = p_i + \sum_{k=0}^{i-2} z_k v, \tag{2.13}$$

for $z \in Z_4$ and $i \in \mathbb{N}$ by (2.4) and (2.12), it is easily seen that the function $\psi_D \circ G \colon Z_4 \to \psi_D(\mathcal{H}(\overline{B}))$ is an imbedding.

Now we show that $\psi_D \circ G(Z_4)$ is a closed subset of $\psi_D(\mathcal{H}(\overline{B}))$. Let $h \in \mathcal{H}(\overline{B})$ be such that there is a sequence $z^1, z^2, \ldots \in Z_4$ with $\lim_{j\to\infty} G_{z^j} \restriction D = h \restriction D$ in $\overline{B}^D$. Since $2u$ and α are elements of D this immediately implies that $h(2u) = 2u$ and $h(\alpha) = \alpha$. Combining the fact that $h(\alpha) = \alpha$ with (2.13) we see that for every $i \in \omega$,

$$\begin{aligned} z_i &= p(h(p_{i+2})) - p(h(p_{i+1})) - (p(p_{i+2}) - p(p_{i+1})) \\ &= \lim_{j\to\infty} \big(p(G_{z^j}(p_{i+2})) - p(G_{z^j}(p_{i+1}))\big) - p(p_{i+2}) + p(p_{i+1}) \\ &= \lim_{j\to\infty} z_i^j \end{aligned}$$

is a well defined real number. Since $z_i^j \in 4^{-i}\omega$ and this set is closed in $\mathbb{R}$ we have $z_i \in 4^{-i}\omega$ for every $i \in \omega$. Note that $h(\mathbf{0}) = \lim_{i\to\infty} h(p_i) \in [0,\infty)v$ because $h(\mathbf{0}) \neq h(\alpha)$. By definition of the z_k's we have $\sum_{k=0}^{i-1} z_k = p(h(p_{i+1})) - p(p_{i+1})$ which means that

$$\|z\|_1 = \lim_{i\to\infty} \big(p(h(p_{i+1})) - p(p_{i+1})\big) = p(h(\mathbf{0})) - p(\mathbf{0}) = p(h(\mathbf{0})) < \infty.$$

Thus $z \in E_4$ and $\lim_{j\to\infty} z^j = z$ in Z_4. This implies that

$$\psi_D(G_z) = \lim_{j\to\infty} \psi_D(G_{z^j}) = \psi_D(h)$$

and the proof is finished. $\square$

For $i \in \omega$ we let $\xi_i \colon E_4 \to E_4$ denote the projection that is given by $\xi_i(z) = (z_0, z_1, \ldots, z_i, 0, 0, \ldots)$.

Proposition 2.4.6. *Let $n \in \mathbb{N} \setminus \{3\}$ and let $O \subset M_n^{n+1}$ be open and not empty. Then there exists a closed imbedding $G \colon E_4 \ni z \to G_z \in \mathcal{H}_O(M_n^{n+1})$, a copy $\hat{\mathbb{R}}_c$ of $\hat{\mathbb{R}}$ in O and a sequence $p_1, p_2, \ldots \in O \setminus \hat{\mathbb{R}}_c$ such that* *(a)* $\lim_{i\to\infty} p_i = 0_{\mathrm{c}} \in \mathbb{R}_{\mathrm{c}}$, *where* $\mathbb{R}_{\mathrm{c}} = \hat{\mathbb{R}}_{\mathrm{c}} \setminus \{\pm\infty_{\mathrm{c}}\}$*;*

(b) *for each* $r \in \hat{\mathbb{R}}_{\mathrm{c}}$ *and* $z \in E_4$ *we have* $G_z(r) = r + \|z\|_1 \in \hat{\mathbb{R}}_{\mathrm{c}}$*;*

(c) *for each* $x \in M_n^{n+1} \setminus \mathbb{R}_{\mathrm{c}}$ *there is an* $i \in \omega$ *such that* $G_z(x) = G_{\xi_i(z)}(x)$ *for every* $z \in E_4$*; and*

(d) $\beta \circ G\colon Z_4 \to \beta(\mathcal{H}(M_n^{n+1}))$ *is a closed imbedding, where* $A = \{\infty_{\mathrm{c}}, p_1, p_2, \ldots\}$ *and* $\beta\colon \mathcal{H}(M_n^{n+1}) \to (M_n^{n+1})^A$ *is given by* $\beta(h) = h{\restriction}A$*.*

The sets $\mathbb{R}_{\mathrm{c}}$ *and* A *can be chosen such that either both consist of interior points of* M_n^{n+1} *or both consist of boundary points of* M_n^{n+1}*. Moreover, for* $n = 1$ *the sets* $\mathbb{R}_{\mathrm{c}}$ *and* A *can be chosen such that* $\mathbb{R}_{\mathrm{c}}$ *consists of interior points of* M_1^2 *and* A *consists of boundary points of* M_1^2*.*

Proof. Take $n \in \mathbb{N} \setminus \{3\}$. First we show that there is a copy $\hat{\mathbb{R}}_{\mathrm{c}}$ of $\hat{\mathbb{R}}$ in $\overline{B}$ and a sequence $p_1, p_2, \ldots$ of elements of $\overline{B}$ such that conditions (a)–(d) with M_n^{n+1} replaced by $\overline{B}$ are satisfied by the closed imbedding $G\colon E_4 \to \mathcal{H}(\overline{B})$ of Lemma 2.4.3. Remember from the construction of $\overline{B}$ in §2.3 that $V = \{0\} \times \mathbb{R} \times \{0\} \times \cdots \times \{0\}$ in $\mathbb{R}^{n+1}$. Let $\hat{\mathbb{R}}_{\mathrm{c}} = V \cup \{\alpha, \beta\} \subset \overline{B}$, where $\alpha = \infty_c$ and $\beta = -\infty_c$, and let $p_i = 2^{-i+2}u$ for $i \in \mathbb{N}$. It is clear that property (a) is satisfied. For property (b), note that it follows from the definition of G, see (2.11), and the fact that the function g from (2.10) is an isometry that

$$G_z(r) = \overline{H}_{g(z)}(r) = r + \|g(z)\|_1 = r + \|z\|_1,$$

for $z \in E_4$ and $r \in \hat{\mathbb{R}}_{\mathrm{c}}$. Property (c) follows immediately from the defintition of G and the construction of $\overline{H}$. See also the remark after equation (2.6). For property (d), note that $A = \{\infty_c, p_1, p_2, \ldots\}$ equals the set C in the definition of the function ψ in Lemma 2.3.1. We see that property (d) follows from the fact that $g\colon Z_4 \to Z_2$ is a closed imbedding and Lemma 2.3.1. We prove the proposition for $n = 1$ and $n \geq 2$ separately.

Case I : $n \in \mathbb{N} \setminus \{1, 3\}$. This is the easy case because, as mentioned before, the set C in the definition of the function ψ in Lemma 2.3.1 is contained in the boundary of the unbounded component of $\mathbb{R}^{n+1} \setminus \overline{B}$. It follows immediately from Lemma 2.4.4 that there is a closed imbedding $G\colon E_4 \to \mathcal{H}_O(M_n^{n+1})$. The proof of this lemma, [7, Remark 4], gives us an imbedding of $\overline{B}$ in M_n^{n+1} and we take the images of $\hat{\mathbb{R}}_c$ and the sequence $p_1, p_2, \dots$ in $\overline{B}$ under this imbedding as the set $\hat{\mathbb{R}}_c$ and the sequence $p_1, p_2, \dots$ in M_n^{n+1}. Then we easily see that G satisfies properties (a)–(c) and we have that $\mathbb{R}_c$ and A both consist of boundary points of M_n^{n+1}. Furthermore, using the same arguments as in the proof of Lemma 2.3.1 we see that G also satisfies property (d).

To show that there is also a suitable imbedding G such that both $\mathbb{R}_c$ and A consist of interior points of M_n^{n+1} we consider two disjoint copies B_1, B_2 of $\overline{B}$ in S^{n+1}. Let U_1 and U_2 be the components of $S^{n+1} \setminus B_1$ and $S^{n+1} \setminus B_2$, respectively, such that ∂U_1 contains the set $\hat{\mathbb{R}}_c \cup \{p_1, p_2, \dots\}$ in B_1 and ∂U_2 contains the set $\hat{\mathbb{R}}_c \cup \{p_1, p_2, \dots\}$ in B_2. Then, using Theorem 2.2.2, we make a new Sierpiński carpet B from B_1 and B_2 by identifying the points of ∂U_1 with the corresponding points on ∂U_2. This means that the set $\hat{\mathbb{R}}_c \cup \{p_1, p_2, \dots\} \subset B_1$ now only contains interior points of B. The imbeddings of E_4 in $\mathcal{H}(B_1)$ and $\mathcal{H}(B_2)$ by Dijkstra naturally give rise to a closed imbedding G of E_4 in $\mathcal{H}(B)$ that satisfies properties (a)–(d) with M_n^{n+1} replaced by B and is such that $\mathbb{R}_c$ and A both consist of interior points of B. Again, it follows from [7, Remark 4] and the proof of Lemma 2.3.1 that there is an imbedding G as in the proposition with the property that $\mathbb{R}_c$ and A both consist of interior points of M_n^{n+1}.

Case II : $n = 1$. We note that in this case the boundary points of $\hat{\mathbb{R}}_c \cup \{p_1, p_2, \dots\}$ in $\overline{B}$, which are all the points of $\hat{\mathbb{R}}_c$ and p_1, lie in the boundary of the unbounded component of $\mathbb{R}^2 \setminus \overline{B}$. This means that we can use the same argument as in the case $n \in \mathbb{N} \setminus \{1, 3\}$ to show that we can find an imbedding G as required and such that $\mathbb{R}_c$ and A both consist of interior points of M_1^2.

Now choose the points p_i in $\overline{B}$ as in the set D in Lemma 2.4.5, so all these points are boundary points of $\overline{B}$. With [7, Remark 4] and the proof of Lemma 2.4.5 we see that there is an imbedding G that is as desired and with the property that $\mathbb{R}_c$ and A both consist of boundary points of M_1^2.

Consider now two disjoint copies B_1 and B_2 of $\overline{B}$ in S^2. We choose the points p_i in B_1 for $i \in \mathbb{N}$ again as in the set D in Lemma 2.4.5.

Let U_1 and U_2 be the components of $S^2 \setminus B_1$ and $S^2 \setminus B_2$, respectively, such that ∂U_1 contains the set $\hat{\mathbb{R}}_c$ in B_1 and ∂U_2 contains the set $\hat{\mathbb{R}}_c$ in B_2. Observe that $\hat{\mathbb{R}}_c$ in B_1 is an arc in the simple closed curve ∂U_1 and similarly, the set $\hat{\mathbb{R}}_c$ in B_2 is an arc in ∂U_2. This means that, using Theorem 2.2.2, we can form a new Sierpiński carpet B from B_1 and B_2 by simply identifying the points of the set $\hat{\mathbb{R}}_c$ in B_1 with the corresponding points of the set $\hat{\mathbb{R}}_c$ in B_2. Then we have that $\mathbb{R}_c \subset B$ now consists of interior points of B and the points $\pm\infty_c$ are boundary points of B. The imbeddings of E_4 in $\mathcal{H}(B_1)$ and $\mathcal{H}(B_2)$ by Dijkstra naturally extend to an imbedding G of E_4 in $\mathcal{H}(B)$ that satisfies the properties (a)–(d) with M_n^{n+1} replaced by B and is such that $\mathbb{R}_c$ consists of interior points of B and A consists of boundary points of B. Using [7, Remark 4] and the proof of Lemma 2.4.5 we see that there exists an imbedding G as in the proposition with the property that $\mathbb{R}_c$ consists of interior points of M_1^2 and A consists of boundary points of M_1^2. $\square$

Together with the following lemma by Dijkstra and van Mill [11, Lemma 10.3] we can prove Theorem 2.4.1.

Lemma 2.4.7. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be continuous. If $g \circ f$ is a closed imbedding then so is f.*

Proof of **Theorem 2.4.1.** Take an open subset U of M_n^{n+1} that contains O. Let ρ be a metric on M_n^{n+1} and let $\hat{\rho}$ be the induced metric on $\mathcal{H}(M_n^{n+1})$: $\hat{\rho}(f,g) = \max_{x\in M_n^{n+1}} \rho(f(x),g(x))$ for $f,g \in \mathcal{H}(M_n^{n+1})$. Note that $\hat{\rho}$ is right-invariant: $\hat{\rho}(f \circ h, g \circ h) = \hat{\rho}(f,g)$ for any $h \in \mathcal{H}(M_n^{n+1})$. We prove the theorem by showing that $\mathcal{H}_U(M_n^{n+1}, D)$ satisfies the conditions of Theorem 2.2.10. Without loss of generality we may assume that $D \cap (M_n^{n+1} \setminus U)$ is dense in $M_n^{n+1} \setminus U$. Let $\mathcal{T}$ be the topology that $\mathcal{H}_U(M_n^{n+1}, D)$ inherits from the zero-dimensional product space D^D via the injection $h \mapsto h{\restriction}D$. It follows from Dijkstra and van Mill [11, Theorem 10.1] that $\mathcal{T}$ is an $F_{\sigma\delta}$-topology that witnesses the almost zero-dimensionality of $\mathcal{H}_U(M_n^{n+1}, D)$.

We select a null sequence of sets $V_0, V_1, \ldots$ such that their closures are disjoint subsets of O. Put $V = \bigcup_{k=0}^{\infty} V_k$. Remember the definitions of the spaces E_4 and Z_4, see (2.9) and the text thereafter, and the definition of the projection ξ_i for $i \in \omega$, see page 37. We let P be the countable dense subset $\bigcup_{i=0}^{\infty} \xi_i(E_4)$ of E_4.

Consider the Cantor set

$$C' = \{z \in E_4 : z_i \in \{0, 4^{-i}\} \text{ for } i \in \omega\},$$

and note that since $\sum_{i=0}^{\infty} 4^{-i} < \infty$ the norm topology and the product topology coincide on C'. Let $\delta\colon C' \to \mathbb{R}^+$ be the imbedding that is given by $\delta(z) = \|z\|_1$. We define $C = \delta(C')$, $\gamma = \delta^{-1}\restriction C$, and $Q = \delta(C' \cap P)$. Thus C is a Cantor set with Q as a countable dense subset and $\|\gamma(r)\|_1 = r$ for each $r \in C$. We define subspaces $\mathcal{E}_{\mathrm{c}}$ and $\mathcal{E}$ of ℓ^1 as follows

$$\mathcal{E}_{\mathrm{c}} = \{z \in \ell^1 : z_i \in C \text{ for } i \in \omega\}$$

and

$$\mathcal{E} = \{z \in \ell^1 : z_i \in Q \text{ for } i \in \omega\}.$$

The subscript c refers to the fact that $\mathcal{E}_{\mathrm{c}}$ is a complete space. In fact, it follows from Corollary 1.2.6 and Theorem 1.2.7 that $\mathcal{E}_{\mathrm{c}}$ is homeomorphic to $\mathfrak{E}_{\mathrm{c}}$. We let Z_{c} and Z stand for $\mathcal{E}_{\mathrm{c}}$ and $\mathcal{E}$, respectively, with the witness topologies that these spaces inherit from $\mathbb{R}^\omega$. Let $\nu\colon \omega \times \omega \to \omega$ be a bijection such that $\nu(i,j) \geq j$ for all $i, j \in \omega$. We define an imbedding $\zeta\colon \mathcal{E}_{\mathrm{c}} \to E_4$ by $(\zeta(z))_{\nu(i,j)} = (\gamma(z_i))_j$ for $z \in \mathcal{E}_{\mathrm{c}}$ and $i, j \in \omega$. It is clear from the definition and the fact that the norm and product topology coincide on the compactum C' that $\zeta\colon Z_{\mathrm{c}} \to Z_4$ is a closed imbedding. Note that $\|\zeta(z)\|_1 = \|z\|_1$ for each $z \in \mathcal{E}_{\mathrm{c}}$, which implies that ζ is also a closed imbedding with respect to the norm topologies (recall that the norm topology is generated by the product topology together with the norm function, see Proposition 1.2.1).

Using Proposition 2.4.6 we can find for every $k \in \omega$ a closed imbedding $G^k\colon E_4 \to \mathcal{H}_{V_k}(M_n^{n+1})$, a copy $\hat{\mathbb{R}}_k$ of $\hat{\mathbb{R}}$ in V_k and a sequence $p_1^k, p_2^k, \ldots$ in $V_k \setminus \hat{\mathbb{R}}_k$ such that the conditions (a)–(d) of Proposition 2.4.6, with $\hat{\mathbb{R}}_{\mathrm{c}}$ replaced by $\hat{\mathbb{R}}_k$ and p_i replaced by p_i^k, are satisfied for G^k. If $x \in \hat{\mathbb{R}}$ we write x_k for the representation of x in $\hat{\mathbb{R}}_k$. Let $A_k = \{\infty_k, p_1^k, p_2^k, \ldots\}$ and let $\beta_k\colon \mathcal{H}(M_n^{n+1}) \to (M_n^{n+1})^{A_k}$ be given by $\beta_k(h) = h\restriction A_k$. Then condition (d) of Proposition 2.4.6 is satisfied for G^k with the set A_k and the map β_k.

We now define $H\colon \mathcal{E}_\mathrm{c} \to \mathcal{H}_V(M_n^{n+1})$ by

$$H_z(x) = \begin{cases} G^0_{\zeta(z)}(x), & \text{if } x \in V_0; \\ G^k_{\gamma(z_{k-1})}(x), & \text{if } x \in V_k \text{ for some } k \in \mathbb{N}; \\ x, & \text{if } x \in M_n^{n+1} \setminus V, \end{cases} \tag{2.14}$$

for $z \in \mathcal{E}_\mathrm{c}$. Since the V_k's form a null sequence it is clear that every H_z is a homeomorphism of M_n^{n+1} and that H_z depends continuously on $z \in \mathcal{E}_\mathrm{c}$. Let $\Pi\colon \mathcal{H}_V(M_n^{n+1}) \to \mathcal{H}_{V_0}(M_n^{n+1})$ be the continuous map that is defined by $\Pi(h) = (h\restriction V_0) \cup e_{M_n^{n+1}\setminus V_0}$. Since ζ and G^0 are closed imbeddings and $\Pi \circ H = G^0 \circ \zeta$ we have by Lemma 2.4.7 that $H\colon \mathcal{E}_\mathrm{c} \to \mathcal{H}_U(M_n^{n+1})$ is also a closed imbedding. Now we consider the three cases of Claim 2.4.2 separately.

Case (i). In this case $D \cap O$ consists entirely of interior points of M_n^{n+1}. Choose a number $k \in \omega$. With Proposition 2.4.6 we can choose the imbedding G^k in (2.14) such that A_k and $\mathbb{R}_k$ consist of interior points of M_n^{n+1}. Note that $\mathbb{R}_k$ is a nowhere dense subset of V_k. This means that we can find a countable dense subset D_k of V_k, consisting of interior points of M_n^{n+1}, with $D_k \cap \mathbb{R}_k = \emptyset$ and $A_k \subset D_k$. Since P is countable and $G^k_z(\mathbb{R}_k) = \mathbb{R}_k$ for all $z \in E_4$, see property (b) of Proposition 2.4.6, we may assume that $G^k_z(D_k) = D_k$ for each $z \in P$. Let $\mathbb{Q}_4$ be the additive group $\{i4^j : i, j \in \mathbb{Z}\}$ and note that $C \cap \mathbb{Q}_4 = Q$. Let $\mathbb{Q}_4^k$ be the copy of $\mathbb{Q}_4$ that lies in $\mathbb{R}_k$, so $\mathbb{Q}_4^k$ consists entirely of interior points of M_n^{n+1}. As observed in Remark 2.2.9 we may assume that the set D has the properties

$$\begin{aligned} D \cap V_0 &= D_0, \\ D \cap V_k &= D_k \cup \mathbb{Q}_4^k \quad \text{for } k \in \mathbb{N}. \end{aligned} \tag{2.15}$$

We verify that

$$\mathcal{E} = \{z \in \mathcal{E}_\mathrm{c} : H_z(D) = D\}$$

and hence that $H\restriction\mathcal{E}$ is a closed imbedding of $\mathcal{E}$ into $\mathcal{H}_U(M_n^{n+1}, D)$ for $n \in \mathbb{N}$. If $H_z \in \mathcal{H}_U(M_n^{n+1}, D)$ and $k \in \mathbb{N}$ then by property (b) of Proposition 2.4.6 we have $H_z(0_k) = \|\gamma(z_{k-1})\|_1 = z_{k-1} \in \mathbb{Q}_4$. Since $z \in \mathcal{E}_\mathrm{c}$ we also have $z_{k-1} \in C$ and hence $z_{k-1} \in Q$. Thus $z \in \mathcal{E}$. Consider now a point $z \in \mathcal{E}$. If $x \in V_k \setminus \mathbb{R}_k$ for some $k \in \omega$ then by property (c) of Proposition 2.4.6 there is a $z' \in P$ such that $H_z(x) = G^k_{z'}(x)$. Since $G^k_{z'}(D_k) = D_k$ we have $x \in D_k = D \cap V_k \setminus \mathbb{R}_k$ if and only if $H_z(x) \in D_k$. Note that $H_z(\mathbb{R}_0) = \mathbb{R}_0$ and that this set is disjoint from D. Consider finally the case that $x \in \mathbb{R}_k$ for $k \in \mathbb{N}$. Then $z_{k-1} \in Q \subset \mathbb{Q}_4$ and $H_z(x) = G^k_{\gamma(z_{k-1})}(x) = x + \|\gamma(z_{k-1})\|_1 = x + z_{k-1}$ which is in $\mathbb{Q}_4$ if and only if $x \in \mathbb{Q}_4$.

Remember that $\mathcal{T}$ is the topology on $\mathcal{H}_U(M^{n+1}_n, D)$ that it inherits from D^D. Let $\mathcal{T}'$ be the topology that $\mathcal{H}(M^{n+1}_n)$ inherits from the product space $(M^{n+1}_n)^D$ and note that $\mathcal{T}'$ restricts to $\mathcal{T}$ on $\mathcal{H}_U(M^{n+1}_n, D)$. We first verify that $H\colon Z_\mathrm{c} \to (\mathcal{H}(M^{n+1}_n), \mathcal{T}')$ is continuous. Let $x \in D$. If $x \notin V$ or if $x \in V_k$ for some $k \in \mathbb{N}$, then $H_z(x)$ depends on at most a single coordinate of z, so continuity with respect to the product topology is obvious. Let $x \in V_0$ and thus $x \in D_0 \subset V_0 \setminus \mathbb{R}_0$. Then by property (c) of Proposition 2.4.6, $G^0_{z'}(x)$ depends on only finitely many coordinates of $z' \in E_4$ and hence $H_z(x) = G^0_{\zeta(z)}(x)$ depends also on only finitely many coordinates of $z \in Z_\mathrm{c}$. This shows that H is continuous with respect to the product topologies. Using property (d) of Proposition 2.4.6 we see that $\beta_0 \circ H = \beta_0 \circ G^0 \circ \zeta$ is a closed imbedding of Z_c into $\beta_0(\mathcal{H}(M^{n+1}_n))$. Since $A_0 \subset D$ we have that $\beta_0\colon (\mathcal{H}(M^{n+1}_n), \mathcal{T}') \to (M^{n+1}_n)^{A_0}$ is continuous. Thus with Lemma 2.4.7 we may conclude that $H\colon Z_\mathrm{c} \to (\mathcal{H}(M^{n+1}_n), \mathcal{T}')$ is a closed imbedding. Since $Z = H^{-1}(\mathcal{H}_U(M^{n+1}_n, D))$ we also see that $H{\restriction}Z$ is a closed imbedding of Z in $(\mathcal{H}_U(M^{n+1}_n, D), \mathcal{T})$.

Consider the point $0_1 \in \mathbb{Q}^1_4 \subset \mathbb{R}_1$. For every $a \in D$ we define the set $\Gamma_a = \{h \in \mathcal{H}_U(M^{n+1}_n, D) : h(0_1) = a\}$. Note that every Γ_a is closed with respect to $\mathcal{T}$ and that $\bigcup_{a \in D} \Gamma_a = \mathcal{H}_U(M^{n+1}_n, D)$. Fix a point $a \in D$ and let $h \in \Gamma_a$. Let $z^i = (4^{-i}, 0, 0, \ldots) \in \mathcal{E}$ for $i \in \mathbb{N}$. Since $\lim_{i\to\infty} z^i = \mathbf{0}$, we have $\lim_{i\to\infty} h \circ H^{-1}_{\mathbf{0}} \circ H_{z^i} = h$ in $\mathcal{H}_U(M^{n+1}_n, D)$. However, $h \circ H^{-1}_{\mathbf{0}} \circ H_{z^i} \notin \Gamma_a$. To see this, note that it follows from Proposition 2.4.6, property (b), that $H_{\mathbf{0}}{\restriction}\hat{\mathbb{R}}_1 = e_{\hat{\mathbb{R}}_1}$ and $H_{z^i}(0_1) = (4^{-i})_1$. This implies that $h(H^{-1}_{\mathbf{0}}(H_{z^i}(0_1))) = h((4^{-i})_1) \neq h(0_1) = a$. Thus Γ_a is nowhere dense in $\mathcal{H}_U(M^{n+1}_n, D)$ and condition (5′) of Theorem 2.2.10 is satisfied.

We now make an observation which will be the key to satisfying conditions (2′) and (4′) of Theorem 2.2.10.

Claim 2.4.8. *If A is an unbounded subset of $\mathcal{E}$ then*

$$\operatorname{diam}_{\hat\rho}\{H_z : z \in A\} \geq \rho(-\infty_0, \infty_0).$$

PROOF. Let $z \in A$ and let $n \in \mathbb{N}$ arbitrary. Select a point $z^n \in A$ such that $\|z^n\|_1 > \|z\|_1 + 2n$. It follows from (2.14), condition (b) of Proposition 2.4.6 and the fact that $\|\zeta(z)\|_1 = \|z\|_1$ for all $z \in \mathcal{E}_{\mathrm{c}}$ that

$$\begin{aligned} H_z\big((-\|z\|_1 - n)_0\big) &= G^0_{\zeta(z)}\big((-\|z\|_1 - n)_0\big) \\ &= -n_0. \end{aligned}$$

Similarly, we see that

$$H_{z^n}\big((-\|z\|_1 - n)_0\big) = (\|z^n\|_1 - \|z\|_1 - n)_0.$$

We conclude that

$$\begin{aligned} \operatorname{diam}_{\hat\rho}\{H_z : z \in A\} &\geq \limsup_{n\to\infty} \hat\rho\big(H_z, H_{z^n}\big) \\ &\geq \lim_{n\to\infty} \rho\big(-n_0, (\|z^n\|_1 - \|z\|_1 - n)_0\big) \\ &= \rho(-\infty_0, \infty_0), \end{aligned}$$

hence (2.4.8) is true. $\diamondsuit$

Let $T = Q^{<\omega}$ and define for $s = q_1 \dots q_k \in T$ with $k \in \omega$ the subspace $\mathcal{E}_s$ of $\mathcal{E}$ by

$$\mathcal{E}_s = \{z \in \mathcal{E} : z_{i-1} = q_i \text{ for } 1 \leq i \leq k\}.$$

With the same arguments as given after Theorem 1.3.5 for $\mathfrak{E}$ we see that with these choices for T and $\mathcal{E}_s$ the conditions of Theorem 1.3.5 are satisfied. Furthermore, these arguments show that every bounded subset of $\mathcal{E}$ is an anchor for $(\mathcal{E}_s)_{s\in T}$ in Z and every nonempty clopen subset of any space $\mathcal{E}_s$ is unbounded. Let $J = \{f_q : q \in Q\}$ be a countable dense subset of $\mathcal{H}_U(M_n^{n+1}, D)$. Since $H\colon Z \to (\mathcal{H}_U(M_n^{n+1}, D), \mathcal{T})$ is a closed map we have that $X_s = \{H_z : z \in \mathcal{E}_s\}$ is closed with respect to $\mathcal{T}$ for each $s \in T$. We define $(E_s)_{s\in T}$ as follows:

$$E_\lambda = X_\lambda \circ J,$$

so $E_\lambda = \{H_z \circ f_q : z \in \mathcal{E}_\lambda \text{ and } q \in Q\}$, and if $s = q_0 \dots q_k \in T \setminus \{\lambda\}$ then

$$E_s = X_{q_1 \dots q_k} \circ f_{q_0}.$$

Note that if $f \in \mathcal{H}_U(M_n^{n+1}, D)$ then the map $h \mapsto h \circ f$ is a homeomorphism of $(\mathcal{H}_U(M_n^{n+1}, D), \mathcal{T})$ as well as of $\mathcal{H}_U(M_n^{n+1}, D)$. So every E_s is closed with respect to $\mathcal{T}$ provided $s \neq \lambda$.

It remains to show that $(E_s)_{s \in T}$ satisfies conditions $(1')$–$(4')$ of Theorem 2.2.10. Since $X_\lambda \neq \emptyset$ we have that E_λ is, just as J, dense in $\mathcal{H}_U(M_n^{n+1}, D)$. The other part of condition $(1')$ follows with the same ease. Since $H \colon \mathcal{E} \to \mathcal{H}_U(M_n^{n+1}, D)$ is an imbedding condition $(3')$ is satisfied. Now let W be any set in $\mathcal{H}_U(M_n^{n+1}, D)$ with $\operatorname{diam}(W) < \rho(-\infty_0, \infty_0)$. We show that W works for condition $(2')$ as well as $(4')$. Let $\sigma = q_0 q_1 \ldots \in [T]$ be such that $E_{\sigma \restriction k} \cap W \neq \emptyset$ for each $k \in \omega$. Putting $\tau = q_1 q_2 \ldots \in [T]$ we have $X_{\tau \restriction k} \cap (W \circ f_{q_0}^{-1}) \neq \emptyset$ for each $k \in \omega$. Since $\hat{\rho}$ is right invariant it follows that $\operatorname{diam}_{\hat{\rho}}(W \circ f_{q_0}^{-1}) < \rho(-\infty_0, \infty_0)$ and hence $P = \{z \in \mathcal{E} : H_z \in W \circ f_{q_0}^{-1}\}$ is bounded by Claim 2.4.8. Thus P is an anchor for $(\mathcal{E}_s)_{s \in T}$ in Z and obviously $\mathcal{E}_{\tau \restriction k} \cap P \neq \emptyset$ for each $k \in \omega$. This means that $\mathcal{E}_{\tau \restriction 0}, \mathcal{E}_{\tau \restriction 1}, \ldots$ converges to an element z in Z. Then $X_{\tau \restriction 0}, X_{\tau \restriction 1}, \ldots$ converges to H_z and $E_{\sigma \restriction 0}, E_{\sigma \restriction 1}, \ldots$ converges to $H_z \circ f_{q_0}$, both with respect to $\mathcal{T}$. We see that condition $(2')$ is satisfied. Now let C be a nonempty clopen subset of some E_s such that $C \subset W$. We may assume that $|s| \geq 1$ and we put $q = s \restriction 1$ and we let the string t be given by the equation $q^\frown t = s$. So $\operatorname{diam}_{\hat{\rho}}(C \circ f_q^{-1}) < \rho(-\infty_0, \infty_0)$ and $C \circ f_q^{-1}$ is a nonempty clopen subset of X_t. This means that $\{z \in \mathcal{E} : H_z \in C \circ f_q^{-1}\}$ is a nonempty, clopen, bounded subset of $\mathcal{E}_t$. This is in contradiction with our earlier observation that nonempty clopen subsets of any space $\mathcal{E}_t$ are unbounded, so condition $(4')$ is satisfied. We conclude that $\mathcal{H}_U(M_n^{n+1}, D)$ satisfies the conditions of Theorem 2.2.10 and hence this space is homeomorphic to $\mathfrak{E}$.

Case (ii). In this case $D \cap \partial U_i \cap O$ is dense in $\partial U_i \cap O$ for every i and the interior points of M_n^{n+1} contained in $D \cap O$ are dense in O. We use the same method as in case (i). Take $k \in \omega$. By means of Proposition 2.4.6 we choose the imbedding G^k in (2.14) again such that the sets A_k and $\mathbb{R}_k$ both consist of interior points of M_n^{n+1}. Noting that $\mathbb{R}_k$ is a nowhere dense subset of M_n^{n+1} we can find a countable dense subset D_k of V_k such that $A_k \subset D_k$, $D_k \cap \mathbb{R}_k = \emptyset$, $D_k \cap \partial U_i$ is dense in $\partial U_i \cap V_k$ for every i with $\partial U_i \cap V_k \neq \emptyset$, and the interior points of M_n^{n+1} in D_k are also dense in V_k. Furthermore, we may assume that $G_z^k(D_k) = D_k$ for each $z \in P$, since P is countable and $G_z^k(\mathbb{R}_k) = \mathbb{R}_k$ for all $z \in E_4$. It follows from Lemma 2.2.8 that we may assume that D has the properties mentioned in (2.15). We continue in precisely the same way as in case (i) to conclude that $\mathcal{H}_U(M_n^{n+1}, D)$ satisfies the conditions of Theorem 2.2.10 which means that it is homeomorphic to $\mathfrak{E}$.

Case (iii). In this case $D \cap \partial U_i \cap O$ is dense in $\partial U_i \cap O$ for every i and $D \cap O$ contains no interior points of M_n^{n+1}. Again, we want D to have the properties as mentioned in (2.15) for appropriate sets D_k so that in the same way as in case (i) (and (ii)) we can conclude that $\mathcal{H}_U(M_n^{n+1}, D)$ is homeomorphic to $\mathfrak{E}$. We have to treat the cases $n = 1$ and $n > 1$ separately.

First we consider the case $n = 1$. We want that $D \cap V_0 = D_0$, with D_0 a countable dense subset of V_0 with $A_0 \subset D_0$ and $D_0 \cap \mathbb{R}_0 = \emptyset$. Since D only contains boundary points of M_1^2 we want that D_0 consists of boundary points of M_1^2. Furthermore, since we are aiming towards Remark 2.2.9 again, we also want that D_0 is dense in $\partial U_i \cap V_0$ for every i with $\partial U_i \cap V_0 \neq \emptyset$. This means that $\mathbb{R}_0$ cannot be contained in the boundary of some component U_i of the complement of M_1^2. Therefore, we choose G^0 in (2.14) such that A_0 consists of boundary points of M_1^2 and $\mathbb{R}_0$ consists of interior points of M_1^2. This is possible according to Proposition 2.4.6. It is clear then that we can find a set D_0 as required and with Remark 2.2.9 we may indeed conclude that $D \cap V_0 = D_0$.

Now take $k \in \mathbb{N}$. Just as in (2.15) we want that $D \cap V_k = \mathbb{Q}_4^k \cup D_k$, where D_k is a countable dense subset of V_k with $D_k \cap \mathbb{R}_k = \emptyset$ and $A_k \subset D_k$. Since D consists entirely of boundary points of M_1^2 we choose G^k in (2.14) such that both A_k and $\mathbb{R}_k$ contain only boundary points of M_1^2. This can be done according to Proposition 2.4.6. Suppose that $\mathbb{R}_k \subset \partial U_{i_k}$ for some component U_{i_k} of the complement of M_1^2. Noting that $\mathbb{R}_k$ is a nowhere dense subset of M_1^2 we can choose the set D_k such that it is made up of boundary points of M_1^2, it is dense in $\partial U_i \cap V_k$ for every $i \in \omega \backslash \{i_k\}$ with $\partial U_i \cap V_k \neq \emptyset$ and it is dense in $(\partial U_{i_k} \backslash \mathbb{R}_k) \cap V_k$. We see that $D_k \cup \mathbb{Q}_4^k$ is a countable dense subset of V_k, consisting entirely of boundary points of M_1^2, that is dense in $\partial U_i \cap V_k$ for every i with

$\partial U_i \cap V_k \neq \emptyset$. It then follows from Remark 2.2.9 that we may assume that indeed $D \cap V_k = \mathbb{Q}_4^k \cup D_k$. We conclude that without loss of generality the set D satisfies the conditions given in (2.15). As before, we may work under the assumption that $G_z^k(D_k) = D_k$ for all $k \in \omega$ and each $z \in P$, so we can continue in the same way as in case (i) to conclude that $\mathcal{H}_U(M_n^{n+1}, D)$ is homeomorphic to $\mathfrak{E}$.

Now consider the case that $n \in \mathbb{N} \setminus \{1, 3\}$. This is easier than the one-dimensional case. Take $k \in \omega$. Using Proposition 2.4.6 we choose the imbedding G^k in (2.14) such that both the sets A_k and $\mathbb{R}_k$ consist of boundary points of M_n^{n+1}. Note that if $\mathbb{R}_k \subset \partial U_{i_k}$ then $\mathbb{R}_k$ is, in contrast to the case $n = 1$, nowhere dense in ∂U_{i_k}. This means that we can find a countable dense subset D_k of V_k, consisting of boundary points of M_n^{n+1}, such that $A_k \subset D_k$, $D_k \cap \mathbb{R}_k = \emptyset$ and $D_k \cap \partial U_i$ is dense in $\partial U_i \cap V_k$ for all i. With Remark 2.2.9 it follows that we may assume that $D \cap V_0 = D_0$ if $k = 0$ and $D \cap V_k = \mathbb{Q}_4^k \cup D_k$ if $k \in \mathbb{N}$, so we may work under the condions given in (2.15). Again, without loss of generality we have $G_z^k(D_k) = D_k$ for all $k \in \omega$ and each $z \in P$, so the same reasoning as in case (i) shows that $\mathcal{H}_U(M_n^{n+1}, D)$ is homeomorphic to $\mathfrak{E}$. $\square$

In analogy to Dijkstra and van Mill [11, Theorem 10.4] and Dijkstra and van Mill [11, Remark 10.7] we can adapt the proof of Theorem 2.4.1 to produce the following slight generalization.

Theorem 2.4.9. *Let X be a locally compact space and let D' be a countable dense subset of X. Suppose that X contains an open subset O' that is homeomorphic to an open set $O \subset M_n^{n+1}$ for some $n \in \mathbb{N} \setminus \{3\}$, such that $D' \cap O'$ corresponds to a countable dense subset D of O that satisfies the conditions of Theorem 2.4.1. Then $\mathcal{H}_U(X, D')$ is homeomorphic to $\mathfrak{E}$ for every open set U that contains O'.*

Chapter 3

Generalized Erdős spaces

3.1 Introduction

In this chapter we will generalize the construction of $\mathfrak{E}$ and $\mathfrak{E}_c$. Before Theorem 1.2.5 we already considered a first generalization by DIJKSTRA [5] as follows. Take $p \geq 1$ and consider the Banach space ℓ^p. If $E_n \subset \mathbb{R}$ for $n \in \omega$ then we let $\mathcal{E} = \{x \in \ell^p \colon x_n \in E_n \text{ for every } n \in \omega\}$ be equipped with the topology that is generated by the p-norm $\|\cdot\|_p$ on ℓ^p. We already stated the next theorem of DIJKSTRA [5, Theorem 1] in Theorem 1.2.5.

Theorem 3.1.1. *Assume that $\mathcal{E}$ is not empty and that every E_n is zero-dimensional. For each $\varepsilon > 0$ we let $\eta(\varepsilon) \in \mathbb{R}^\omega$ be given by*

$$\eta(\varepsilon)_n = \sup\{|a| : a \in E_n \cap [-\varepsilon, \varepsilon]\},$$

where $\sup \emptyset = 0$. The following statements are equivalent:

(1) $\|\eta(\varepsilon)\|_p = \infty$ *for each* $\varepsilon > 0$*;*

(2) *there exists an* $x \in \prod_{n=0}^{\infty} E_n$ *with* $\|x\|_p = \infty$ *and* $\lim_{n\to\infty} x_n = 0$*;*

(3) *every nonempty clopen subset of* $\mathcal{E}$ *is unbounded;*

(4) $\mathcal{E}$ *is cohesive; and*

(5) $\dim \mathcal{E} > 0$*.*

Furthermore, Dijkstra and van Mill [10, Theorem 37] proved the following theorem about Polishable F_σ-ideals on ω (definitions can be found in §3.2).

Theorem 3.1.2. *Suppose that $\mathcal{I}$ is a Polishable F_σ-ideal on ω and let φ be an LSC submeasure on ω with $\mathcal{I} = \mathrm{Exh}(\varphi) = \mathrm{Fin}(\varphi)$. Let τ_d be the Polish topology on $\mathcal{I}$ that is generated by the metric $d(X,Y) = \varphi(X \triangle Y)$ for $X, Y \in \mathcal{I}$. Then the following statements are equivalent:*

(1) *for every $\varepsilon > 0$ we have $\{n \in \omega : \varphi(\{n\}) \leq \varepsilon\} \notin \mathcal{I}$;*

(2) *there is a $B \in \mathcal{P}(\omega) \setminus \mathcal{I}$ with $\lim_{n\to\infty} \varphi(\{n\} \cap B) = 0$;*

(3) $\dim(\mathcal{I}, \tau_d) > 0$*; and*

(4) $(\mathcal{I}, \tau_d)$ *is homeomorphic to $\mathfrak{E}_{\mathrm{c}}$.*

In fact, [10, Theorem 37] contains more equivalent statements. However, these are not relevant for the discussion in this chapter. A more complete version of [10, Theorem 37] can be found in Theorem 4.1.1. Here we note the result of Solecki [28] that an ideal $\mathcal{I}$ on ω is Polishable and F_σ if and only if $\mathcal{I} = \mathrm{Exh}(\varphi) = \mathrm{Fin}(\varphi)$ for some LSC submeasure φ on ω, see Theorem 3.2.4.

Observe that the two mentioned theorems have an analogous structure but that the spaces that are the subject of the theorems seem very different. Our goal in this chapter is to explain the connection between these theorems by introducing a general setting in which we shall derive Theorem 3.4.7, our main result, which contains Theorem 3.1.1 as well as the equivalence of statements (1) to (3) of Theorem 3.1.2 as special cases. Theorem 3.4.7 also shows that we may add the statement that every nonempty clopen subset of $\mathcal{I}$ is φ-unbounded, and hence also the statement that $\mathcal{I}$ is cohesive, to the statements mentioned in Theorem 3.1.2. A subset A of ω is called *φ-bounded* if there is an $M \in \mathbb{N}$ such that $\varphi(a) \leq M$ for all $a \in A$, and *φ-unbounded* otherwise. Adding these statements to Theorem 3.1.2 makes the analogy with Theorem 3.1.1 even more clear.

In our general setting we study a space $\mathcal{E}$, see (3.5), that is a generalization of both the $\mathcal{E}$-space mentioned in Theorem 3.1.1 and the space $\mathcal{I}$ in Theorem 3.1.2: it is the generalized Erdős space to which we refer in the title of this chapter. Motivated by statement (4) of Theorem 3.1.2 and by Dijkstra and van Mill [10, Theorem 23] we present in §3.5 sufficient conditions under which the space $\mathcal{E}$ is homeomorphic to $\mathfrak{E}_\mathrm{c}$. As a result we derive a full generalization of Theorem 3.1.2. In §3.6 we give sufficient conditions under which the space $\mathcal{E}$ is homeomorphic to $\mathfrak{E}$, generalizing Dijkstra and van Mill [11, Proposition 8.26]. In §3.7 we introduce the concept of a fixed point and show that under certain conditions we have a natural one-point connectification of $\mathcal{E}$ such that the added point must be a fixed point. This generalizes Abry, Dijkstra and van Mill [2, Theorem 16]. *Unless stated otherwise in this chapter every space is separable and metrizable.*

3.2 Preliminaries

Remember that $\mathbb{R}^+ = [0,\infty)$ and let $\tilde{\mathbb{R}} = (-\infty,\infty]$. We consider the topological group $\mathcal{P}(\omega)$, see §1.1.2. Let τ_w be the product topology on 2^ω. The reason for the subscript 'w', which denotes the fact that we view the product topology as a weak topology on 2^ω, is explained in §3.3. We shall simply denote the topology that a subset of 2^ω inherits from $(2^\omega,\tau_\mathrm{w})$ by τ_w as well. Remember that an *ideal* I on ω is a subset of $\mathcal{P}(\omega)$ such that I contains the finite sets, $B \in I$ whenever $B \subset A \in I$, and $A \cup B \in I$ whenever $A, B \in I$. An ideal is clearly a subgroup of 2^ω. Of particular interest are the *Polishable ideals* on ω.

Definition 3.2.1. An ideal I is *Polishable* if there exists a Polish group topology τ on I such that the family of Borel sets with respect to τ is equal to the family of Borel sets of I with respect to τ_w.

It has been shown that if such a Polish topology exists, then it is unique, see Kechris [22, Theorem 9.10]. We make the following useful observation.

Proposition 3.2.2. *An ideal I is Polishable if and only if there exists a Polish group topology on I that is stronger than the product topology τ_w. In this case this Polish group topology is the unique topology as mentioned after Definition 3.2.1.*

Proof. '$\Rightarrow$'. Suppose that I is Polishable. This means that there is a Polish group topology τ on I generating the same Borel sets as τ_w.

We show that τ is necessarily stronger than τ_{w}. Consider the identity function $e_I\colon (I,\tau) \to (I,\tau_{\mathrm{w}})$. Note that (I,τ) and (I,τ_{w}) are both topological groups and e_I is a homomorphism. Furthermore, (I,τ) is a Polish space and hence a Baire space and since τ and τ_{w} generate the same σ-algebra we have that e_I is Borel and therefore Baire measurable. We see that we can apply KECHRIS [22, Theorem 9.10] to conclude that e_I is continuous, which means that $\tau \supset \tau_{\mathrm{w}}$.

'$\Leftarrow$'. Suppose there is a Polish group topology τ on I that is stronger than τ_{w}. We want to show that the Borel σ-algebra $\mathcal{B}(\tau)$ on I generated by τ is the same as the Borel σ-algebra $\mathcal{B}(\tau_{\mathrm{w}})$ on I generated by τ_{w}. Since $\tau_{\mathrm{w}} \subset \tau$ we immediately see that $\mathcal{B}(\tau_{\mathrm{w}}) \subset \mathcal{B}(\tau)$, so it is left to show that $\mathcal{B}(\tau) \subset \mathcal{B}(\tau_{\mathrm{w}})$. Consider the identity map $e_I\colon (I,\tau) \to (\overline{I},\tau_{\mathrm{w}})$, where $\overline{I}$ denotes the closure of I in 2^ω. Since $\tau \supset \tau_{\mathrm{w}}$ we know that e_I is continuous. It follows from KECHRIS [22, Theorem 15.1] that any set $A \in \mathcal{B}(\tau)$ is a Borel set in $\overline{I}$ with the product topology which implies that $A \in \mathcal{B}(\tau_{\mathrm{w}})$. This shows that $\mathcal{B}(\tau) \subset \mathcal{B}(\tau_{\mathrm{w}})$. $\square$

The following two theorems are taken from SOLECKI [28, 29].

Theorem 3.2.3. *If φ is an LSC submeasure on ω then*

$$d(A,B) = \varphi(A \mathbin{\triangle} B) \text{ for } A, B \subset \omega$$

restricts to an invariant, complete, separable metric on $\mathrm{Exh}(\varphi)$.

Observe that the group topology τ_d on $I = \mathrm{Exh}(\varphi)$ generated by d is stronger than τ_{w}. Using Proposition 3.2.2 we see that I is a Polishable ideal and τ_d is the unique Polish group topology on I that satisfies the condition of Definition 3.2.1. Note that in general the d-topology on $\mathrm{Fin}(\varphi)$ may be nonseparable. The following theorem contains another characterization of Polishable ideals.

Theorem 3.2.4. *Let I be an ideal on ω. Then the following statements hold* (*where φ stands for an LSC submeasure on ω*):

(1) *I is Polishable $\Leftrightarrow$ $I = \mathrm{Exh}(\varphi)$ for some finite φ;*

(2) *I is F_σ in 2^ω $\Leftrightarrow$ $I = \mathrm{Fin}(\varphi)$ for some φ; and*

(3) *I is Polishable and F_σ $\Leftrightarrow$ $I = \mathrm{Exh}(\varphi) = \mathrm{Fin}(\varphi)$ for some φ.*

3.3 Generalized Erdős type spaces

In this section we introduce the setting for Theorem 3.4.7 which generalizes Theorem 3.1.1 and the equivalence of the statements (1)–(3) of Theorem 3.1.2.

An obvious resemblance between Theorem 3.1.1 and Theorem 3.1.2 is that both theorems deal with an LSC function on a product space: in Theorem 3.1.1 we have the p-norm on $\tilde{\mathbb{R}}^\omega$ and in Theorem 3.1.2 we have an LSC submeasure φ on 2^ω. Our first step is to generalize the underlying product space to a product of arbitrary spaces with a reflexive relation. So suppose that (X, R) is a pair consisting of a topological space X and a relation R in X such that for all $x \in X$ we have:

$$xRx,$$

that is, R is *reflexive*. Note that at this point we do not assume any connection between the topology on X and the relation R in X.

Definition 3.3.1. Let $A \subset X$.

(i) A point $a \in A$ is a *least* element of A in (X, R) if aRx for every $x \in A$.

(ii) A point $a \in A$ is a *greatest* element of A in (X, R) if xRa for every $x \in A$.

Note that it is possible that A has more than one least element and more than one greatest element.

Definition 3.3.2. Let $A \subset X$.

(i) A point $x \in X$ is an *upper bound* of A in (X, R) if aRx for all $a \in A$.

(ii) A point $x \in X$ is called a *supremum* of A in (X, R) if it is a least element of the set of all upper bounds of A in (X, R).

Note that the difference between a greatest element of A and an upper bound of A is that the latter is not required to be an element of A. Again, a set can have many upper bounds and suprema. We use the notation $\operatorname{Sup} A$ or $\operatorname{Sup}_R A$ to denote the *set* of suprema of A with respect to the relation R in X. If R is an ordering of X then a set has at most one least (and greatest) element, and hence at most one supremum. In this case we use the notation $\sup A$ or $\sup_R A$ for the supremum of A, if it exists, and we have

$$x \in \operatorname{Sup} A \Longleftrightarrow x = \sup A,$$

for every subset A of X with a supremum. Let for $n \in \omega$ the pair (X_n, R_n) consist of a topological space X_n and a reflexive relation R_n in X_n, where, again, at this point we do not assume any connection between the topology on X_n and the relation R_n in X_n.

We assume throughout that $\operatorname{Sup}_{R_n} A \neq \emptyset$ *for every subset* A *of* X_n.

In particular we have for every $n \in \omega$ that $\operatorname{Sup}_{R_n} \emptyset \neq \emptyset$ and $\operatorname{Sup}_{R_n} X_n \neq \emptyset$, which means that every space X_n has a least element and a greatest element. For $n \in \omega$ put

$$L_n = \operatorname{Sup}_{R_n} \emptyset = \{x \in X_n : \ x \text{ is a least element of } X_n\}.$$

For the sake of convenience we choose for every $n \in \omega$ an $l_n \in L_n$ and simply denote it by 0. So we get the vector $\underline{0} = (0, 0, \ldots) \in L = \prod_{n=0}^{\infty} L_n$.

For example, in the case of Theorem 3.1.1 we take for every $n \in \omega$ the pair $\big(X_n, R_n\big) = \big(\tilde{\mathbb{R}}, \tilde{R}\big)$, where $\tilde{R}$ is the relation in $\tilde{\mathbb{R}}$ given by

$$x\tilde{R}y \Longleftrightarrow |x| \leq |y|. \tag{3.1}$$

Clearly, $\tilde{R}$ is reflexive and 0 is the unique least element of $\tilde{\mathbb{R}}$ in $(\tilde{\mathbb{R}}, \tilde{R})$, so $L_n = \{0\}$ for all $n \in \omega$. Take a subset A of $\tilde{\mathbb{R}}$ and define $s = \sup\{|a| : a \in A\}$, with $\sup \emptyset = 0$, where the supremum is taken with respect to the natural ordering of $\tilde{\mathbb{R}}$. It is not difficult to show that

$$\operatorname{Sup}_{\tilde{R}} A = \begin{cases} \{-s, s\}, & \text{if } s \neq \infty; \\ \{s\}, & \text{if } s = \infty, \end{cases} \tag{3.2}$$

which means that $\operatorname{Sup}_{\tilde{R}} A \neq \emptyset$. In Theorem 3.1.2 we are dealing with an LSC submeasure on $2^\omega = \{0, 1\}^\omega$. We take $X_n = \{0, 1\}$ and we let

R_n be the natural ordering '$\leq$' for all $n \in \omega$. Obviously, $\mathrm{Sup}_{\leq} A = \{\sup_{\leq} A\} \neq \emptyset$ for every $A \subset \{0,1\}$ (note that $\sup \emptyset = 0$) and $L_n = \{0\}$ for every $n \in \omega$.

Returning to the general case, let τ_{w} denote the product topology on $X = \prod_{n=0}^{\infty} X_n$ and define a relation R in X by

$$xRy \iff \forall\, n \in \omega : x_n R_n y_n. \tag{3.3}$$

Note that R is reflexive and that $L = \mathrm{Sup}_R \emptyset$. Furthermore, let for $n \in \omega$ the function $\xi_n \colon X \to X$ be the projection

$$\xi_n(z) = \begin{cases} (z_0, \ldots, z_{n-1}, 0, 0, \ldots), & \text{if } n \geq 1; \\ \underline{0}\,, & \text{if } n = 0, \end{cases}$$

and let $\zeta_n \colon X \to X$ be given by

$$\zeta_n(z) = \begin{cases} (0, \ldots, 0, z_n, z_{n+1}, \ldots), & \text{if } n \geq 1; \\ z, & \text{if } n = 0. \end{cases}$$

Now that we have generalized the underlying product space of Theorem 3.1.1 and Theorem 3.1.2 to the space X, we introduce an LSC function on X that generalizes the p-norm in Theorem 3.1.1 and the submeasure in Theorem 3.1.2. Consider an LSC function $\chi \colon X \to [0, \infty]$ such that:

(a) $\chi(x) = 0$ for all $x \in L$;

(b) (Monotonicity) if xRy then $\chi(x) \leq \chi(y)$ for all $x, y \in X$;

(c) (Subadditivity) for all $x \in X$ and all $n \in \omega$ we have

$$\chi(x) \leq \chi\big(\xi_n(x)\big) + \chi\big(\zeta_n(x)\big), \text{ and}$$

(d) if $\emptyset \neq A_n \subset X_n$ for all $n \in \omega$ and $a^s \in \prod_{n=0}^{\infty} \mathrm{Sup}_{R_n} A_n$, then

$$\chi(a^s) \leq \sup\left\{ \chi(a) : a \in \prod_{n=0}^{\infty} A_n \right\}.$$

Of course, the last supremum is the ordinary supremum in $[0,\infty]$ with respect to the natural order. Note that in condition (d) we actually have

$$\chi(a^s) = \sup\left\{\chi(a) : a \in \prod_{n=0}^{\infty} A_n\right\} \tag{3.4}$$

because of condition (b). If we compare property (d) with the characterization of lower semi-continuity given after Definition 1.1.8 in Chapter 1 we can, loosely speaking, interpret it as a kind of LSC property of χ with respect to the relation R in X.

The following lemma shows that the choice of $0 \in L_n$ in the definition of the functions ξ_i and ζ_i is irrelevant for property (c) of χ.

Lemma 3.3.3. *Let $x, y \in X$ be such that $x_n = y_n$ or $x_n, y_n \in L_n$ for all $n \in \omega$. Then $\chi(x) = \chi(y)$.*

PROOF. It follows easily that $x_n R_n y_n$ and $y_n R_n x_n$ for all n. This means that xRy and yRx, so using the monotonicity of χ we conclude that $\chi(x) = \chi(y)$. $\square$

For example, it follows that all points in $L = \mathrm{Sup}_R\,\emptyset$ have the same χ-value, which means that property (a) of χ is equivalent to saying that $\chi(\underline{0}) = 0$.

We now show that the p-norm on $\tilde{\mathbb{R}}^\omega$ for $p \geq 1$ and an LSC submeasure φ on 2^ω are special cases of such a χ. Let $p \geq 1$ and consider the p-norm on $X = \tilde{\mathbb{R}}^\omega$. We already noted that the p-norm is an LSC function on $\tilde{\mathbb{R}}^\omega$ (with the product topology). Using (3.1), the relation R in $\tilde{\mathbb{R}}^\omega$ defined in (3.3), now becomes

$$xRy \iff \forall n \in \omega,\ |x_n| \leq |y_n|.$$

Claim 3.3.4. *The p-norm $\|\cdot\|_p$ satisfies properties (a) through (d).*

PROOF. We already noted that $L = \{(0, 0, \ldots)\}$ from which it is clear that property (a) is satisfied. If $x, y \in \tilde{\mathbb{R}}^\omega$ are such that xRy then we have $|x_n| \leq |y_n|$ for all $n \in \omega$. This implies that $\|x\|_p \leq \|y\|_p$, so property (b) is satisfied. For property (c) note that for all $x \in \tilde{\mathbb{R}}^\omega$ and all $k \in \omega$ we have $x = \xi_k(x) + \zeta_k(x)$ and that the norm function satisfies the triangle inequality on $\tilde{\mathbb{R}}^\omega$. We see that $\|x\|_p \leq \|\xi_k(x)\|_p + \|\zeta_k(x)\|_p$

and property (c) is satisfied. For property (d) we take for every $n \in \omega$ a nonempty subset A_n of $\tilde{\mathbb{R}}$ and a point $a^s \in \prod_{n=0}^{\infty} \operatorname{Sup}_{\tilde{R}} A_n$. From (3.2) we know that $|a^s_n| = \sup\{|a| : a \in A_n\}$ for all n. This means that for all $n \in \omega$ we can find a sequence $(a^m_n)_{m\in\omega}$ of elements in A_n such that $\lim_{m\to\infty} |a^m_n| = |a^s_n|$. Define for $m \in \omega$ the point $a^m = (a^m_0, a^m_1, \ldots) \in \prod_{n=0}^{\infty} A_n$. We see that $\lim_{m\to\infty} |a^m| = |a^s|$ in $\tilde{\mathbb{R}}$, where $|a^m|_n = |a^m_n|$ and $|a^s|_n = |a^s_n|$ for all $n \in \omega$. With the lower semi-continuity of $\|\cdot\|_p$ we find

$$\|a^s\|_p \leq \liminf_{m\to\infty} \|a^m\|_p \leq \sup\left\{ \|a\|_p : a \in \prod_{n=0}^{\infty} A_n \right\}.$$

We conclude that property (d) is satisfied. ♢

Now consider an LSC submeasure φ on $X = \{0,1\}^\omega$. Since we chose the natural ordering R_n on $X_n = \{0,1\}$, we see that the resulting relation R on $X = 2^\omega$ corresponds to the inclusion relation on $\mathcal{P}(\omega)$.

Claim 3.3.5. *Let φ be an LSC submeasure on ω. Then φ satisfies the properties* (a) *through* (d).

Proof. We already noted that in this case we have $L_n = \{0\}$ for all $n \in \omega$, so property (a) just says that $\varphi(\emptyset) = 0$, which is true. For property (b) and (c), let x correspond to the subset A of ω and y to the subset B of ω. Then property (b) says that if $A \subset B$ then $\varphi(A) \leq \varphi(B)$, which is also true. Note that $\xi_k(x)$ denotes the set $A \cap k$ and $\zeta_k(x)$ denotes the set $A \setminus k$, so property (c) says that $\varphi(A) \leq \varphi(A \cap k) + \varphi(A \setminus k)$, which follows from the subadditivity of φ. For property (d), just note that $a^s \in \prod_{n=0}^{\infty} A_n$. ♢

Now that we have generalized the setting of Theorem 3.1.1 and Theorem 3.1.2 to an LSC function χ on X, satisfying properties (a) to (d), we want to generalize the construction of the spaces studied in these theorems. Theorem 3.1.2 deals with the space $I = \operatorname{Exh}(\varphi) = \operatorname{Fin}(\varphi)$. First we generalize the definitions of $\operatorname{Exh}(\varphi)$ and $\operatorname{Fin}(\varphi)$ for a submeasure φ on ω as given in Chapter 1 to the function χ:

$$\begin{aligned} \operatorname{Exh}(\chi) &= \{x \in X : \lim_{m\to\infty} \chi\big(\zeta_m(x)\big) = 0 \text{ in } \mathbb{R}\}, \\ \operatorname{Fin}(\chi) &= \{x \in X : \chi(x) < \infty\}. \end{aligned}$$

Remark 3.3.6. The addition 'in $\mathbb{R}$' means that the entire sequence lies in $\mathbb{R}$, that is, $\chi(\zeta_m(x)) < \infty$ for every $m \in \omega$. This guarantees that as with submeasures we always have $\mathrm{Exh}(\chi) \subset \mathrm{Fin}(\chi)$ (see Lemma 1.1.5).

Note that it follows from Lemma 3.3.3 that $\mathrm{Exh}(\chi)$ is independent of the choice of $\underline{0} \in L$.

For the remainder of this chapter let $(E_n)_{n\in\omega}$ be a fixed sequence of sets such that $E_n \subset X_n$ for all n. We define

$$\mathcal{E} = \mathrm{Fin}(\chi) \cap \prod_{n=0}^{\infty} E_n. \tag{3.5}$$

If we take χ equal to the p-norm $\|\cdot\|_p$ on $\tilde{\mathbb{R}}^\omega$ for some $p \geq 1$ then we have $\mathrm{Fin}(\chi) = \ell^p = \mathrm{Exh}(\chi)$. We see that $\mathcal{E}$ defined by formula (3.5) corresponds to the $\mathcal{E}$-space in Theorem 3.1.1.

It is easily seen that the definitions of $\mathrm{Exh}(\chi)$ and $\mathrm{Fin}(\chi)$ reduce to those given in §1.1.2 if we take $E_n = X_n = \{0, 1\}$ and if χ is an LSC submeasure on ω, see also (1.1) on page 5. In Theorem 3.1.2 the assumption is $\mathrm{Fin}(\chi) = \mathrm{Exh}(\chi)$ and the theorem is known to be false without this assumption; see Dijkstra and van Mill [10, Example 42]. Therefore it is not unexpected that for the general theorems we will also need the assumption

$$\mathrm{Fin}(\chi) = \mathrm{Exh}(\chi).$$

It will be useful to consider the injections $\alpha_n : X_n \to X$ defined by

$$\big(\alpha_n(x)\big)_j = \begin{cases} x, & \text{if } j = n; \\ 0, & \text{if } j \neq n, \end{cases}$$

for $n \in \omega$ and $x \in X_n$. For $n \in \omega$ we define the function $\chi_n \colon X \to [0, \infty]$ by

$$\chi_n(x) = \chi(\alpha_n(x_n)).$$

Lemma 3.3.3 tells us that the value of χ_n is independent of the choice of $0 \in L_j$ in the definition of α_n.

Remark 3.3.7. Note that if χ is the p-norm and $x \in \ell^p$, then $\chi_n(x) = |x_n|$; and that if χ is a submeasure φ on ω and $A \subset \omega$, then $\chi_n(A) = \varphi(\{n\} \cap A)$.

For the following lemma, which is a generalization of Lemma 1.1.10, we note that the lower semi-continuity and monotonicity of χ easily imply that

$$\forall x \in X,\ \lim_{n\to\infty} \chi\big(\xi_n(x)\big) = \chi(x). \tag{3.6}$$

Lemma 3.3.8. *Let $x \in X$. Then we have the following inequalities:*

$$\sup_{n\in\omega} \chi_n(x) \le \chi(x) \le \sum_{n=0}^{\infty} \chi_n(x).$$

Proof. The left-hand inequality follows from monotonicity. For the other inequality we first show that for $x \in X$ and $m \in \omega$ we have

$$\chi(\xi_m(x)) \le \sum_{n=0}^{m-1} \chi_n(x). \tag{3.7}$$

We prove this by induction on m. If $m = 0$ then $\xi_m(x) = \underline{0}$ and hence $\chi(\xi_m(x)) = 0$ by property (a) of χ. On the right hand side we have an empty sum which is zero by definition.

For the induction step suppose that (3.7) is true for some $m \in \omega$. By the subadditivity (property (c)) of χ we have

$$\chi(\xi_{m+1}(x)) \le \chi(\xi_m(x)) + \chi(\alpha_m(x_m)) \le \sum_{n=0}^{m} \chi_n(x).$$

This completes the induction. Now take an arbitrary $x \in X$. By (3.6) and (3.7) it follows that

$$\chi(x) = \lim_{m\to\infty} \chi\big(\xi_m(x)\big) \le \lim_{m\to\infty} \sum_{n=0}^{m-1} \chi_n(x) = \sum_{n=0}^{\infty} \chi_n(x). \tag{3.8}$$

This proves the lemma. $\square$

Lemma 3.3.9. *Let $x \in \mathrm{Fin}(\chi)$ and let $y \in X$ be such that there is a finite set $A \subset \omega$ with $x_i = y_i$ for each $i \in \omega \setminus A$. If $\chi_i(y) < \infty$ for each $i \in A$, then $y \in \mathrm{Fin}(\chi)$.*

Proof. Let $k \in \omega$ be such that $A \subset k$. Note that by Lemma 3.3.8, monotonicity, and subadditivity we have

$$\chi(y) \leq \sum_{i=0}^{k-1} \chi_i(y) + \chi(\zeta_k(x)) \leq \sum_{i \in A} \chi_i(y) + (k+1)\chi(x) < \infty.$$

Thus $y \in \mathrm{Fin}(\chi)$. $\square$

Now we address the question which topology to take on $\mathcal{E}$. Since we are trying to generalize Theorem 3.1.1 and Theorem 3.1.2, we want a topology on $\mathcal{E}$ that reduces to the norm topology if we take χ to be the p-norm on $\tilde{\mathbb{R}}^\omega$, and that reduces to the τ_d topology if we take χ to be the LSC submeasure φ on ω as mentioned in Theorem 3.1.2. Remember that τ_d is the topology on $\mathrm{Exh}(\varphi)$ that is generated by the metric d mentioned in Theorem 3.2.3. It is clear that we cannot generalize the construction of the τ_d topology on I in Theorem 3.1.2 to our space $\mathcal{E}$. However, as mentioned in §1.2, the norm topology on ℓ^p is the weakest topology that contains the product topology inherited from $\tilde{\mathbb{R}}^\omega$ and that makes the norm continuous (the Kadec property). This construction of the norm topology can easily be generalized to our general setting. Remember that we have the product topology τ_{w} on X.

Definition 3.3.10. Let τ_χ be the weakest topology on X that contains τ_{w} and that makes χ continuous.

Since τ_χ is a stronger topology than τ_{w}, we will also refer to τ_{w} as the weak topology, which explains the subscript 'w'. Note that $\mathcal{B}$ given by

$$\mathcal{B} = \{O \cap \chi^{-1}([0,t)) : O \in \tau_{\mathrm{w}} \text{ and } t \in (0,\infty)\} \cup \{V : V \in \tau_{\mathrm{w}}\}$$

forms a basis for τ_χ. Since χ is LSC it is easily seen that τ_χ is a regular topology. Consequently, τ_χ is separable and metrizable by the Urysohn Metrization Theorem. We endow $\mathcal{E}$ with the subspace topology τ_χ it inherits from $\big(X, \tau_\chi\big)$. Since $\mathcal{E} \subset \mathrm{Fin}(\chi)$ we see that $\mathcal{C}$ given by

$$\mathcal{C} = \{O \cap \chi^{-1}([0,t)) : O \in \tau_{\mathrm{w}} \text{ on } \mathcal{E} \text{ and } t \in (0,\infty)\} \tag{3.9}$$

forms a basis for τ_χ on $\mathcal{E}$. From now on we shall simply denote the topology that a subset of X inherits from (X, τ_{w}) by τ_{w} and the topology it inherits from (X, τ_χ) by τ_χ.

The previous observations imply that our general space $\mathcal{E}$ can indeed be reduced to the space $\mathcal{E}$ of Theorem 3.1.1. To get the ideal I of Theorem 3.1.2 we take $(X_n, R_n) = (\{0,1\}, \leq)$, $\chi = \varphi$, the LSC submeasure of the theorem and $E_n = \{0,1\}$ for all $n \in \omega$. However, it follows from Dijkstra and van Mill [10, Remark 41] that the topology $\tau_\varphi = \tau_\chi$ on I need not be equal to τ_d, the Polish topology on I in Theorem 3.1.2. The following lemma gives the relation between the topologies $\tau_{\mathrm{w}}, \tau_\varphi$ and τ_d on $\mathrm{Exh}(\varphi)$ and again justifies our terminology of 'the weak topology' for the product topology τ_{w}.

Lemma 3.3.11. *Let φ be an LSC submeasure on ω. Then we have the following relation between the topologies τ_{w}, τ_φ and τ_d on* $\mathrm{Exh}(\varphi)$*: $\tau_{\mathrm{w}} \subset \tau_\varphi \subset \tau_d$.*

Proof. The inclusion $\tau_{\mathrm{w}} \subset \tau_\varphi$ follows immediately from the definition of τ_φ. Because $\varphi(\{n\}) > 0$ for all $n \in \omega$ it follows that $\tau_d \supset \tau_{\mathrm{w}}$ (see also the observation following Theorem 3.2.3). Furthermore, using the monotonicity and subadditivity of φ it is not difficult to show that

$$|\varphi(A) - \varphi(B)| \leq d(A, B)$$

for all $A, B \in \mathrm{Exh}(\varphi)$, which means that φ is continuous with respect to τ_d. Since τ_φ is the weakest topology that makes φ continuous and that contains τ_{w}, we have $\tau_\varphi \subset \tau_d$. □

So we are interested in LSC submeasures φ on ω with $\tau_\varphi = \tau_d$ on $\mathrm{Exh}(\varphi)$. In analogy to the definition of Kadec norms on ℓ^p given in §1.2 we call these submeasures *Kadec submeasures.*

Definition 3.3.12. An LSC submeasure φ on ω is called a *Kadec submeasure* if τ_φ equals the Polish group topology τ_d on $\mathrm{Exh}(\varphi)$.

In Definition 4.3.7 we define a Kadec submeasure on an arbitrary cardinal number in the same way. We have the following result.

Proposition 3.3.13. *An LSC measure φ on ω is a Kadec submeasure.*

Proof. We know from Lemma 3.3.11 that $\tau_\varphi \subset \tau_d$ on $\mathrm{Exh}(\varphi)$ so it suffices to show that $\tau_d \subset \tau_\varphi$. Take $X \in \mathrm{Exh}(\varphi)$ and let $\varepsilon > 0$. We show that $B_d(X, \varepsilon)$, the ε-ball with respect to the metric d as in Theorem 3.2.3 with center X, is a neighbourhood of X in τ_φ. Since $X \in \mathrm{Exh}(\varphi)$ there is a finite subset F of X such that $\varphi(F) > \varphi(X) - \varepsilon/3$. Now define $O \in \tau_\varphi$ as

$$O = \{Y \in \mathrm{Exh}(\varphi) : \varphi(Y) < \varphi(X) + \varepsilon/3 \text{ and } Y \supset F\}.$$

It is clear that $X \in O$. Since φ is a measure we have for every $Y \in \mathrm{Exh}(\varphi)$ that

$$d(X, Y) = \varphi(X \triangle Y) = \varphi(X) + \varphi(Y) - 2\varphi(X \cap Y).$$

Take Y in O. Then $\varphi(X \cap Y) \geq \varphi(F) > \varphi(X) - \varepsilon/3$ and with the above equality we find that

$$d(X, Y) < \varphi(X) + \varphi(X) + \varepsilon/3 - 2\varphi(X) + 2\varepsilon/3 = \varepsilon.$$

This means that $X \in O \subset B_d(X, \varepsilon)$. $\square$

Let for $A \subset \omega$ the function $\zeta_A \colon 2^\omega \to 2^\omega$ be given by $\zeta_A(X) = X \setminus A$. In line with the remark after equation (1.1) we write ζ_k for $\zeta_{\{0,\ldots,k-1\}}$ (where $\{0, \ldots, k-1\}$ equals the empty set if $k = 0$), which is consistent with the definition of ζ_k on page 55. We can now state the following characterization for Kadec submeasures. For a more general version see Proposition 4.3.9.

Proposition 3.3.14. *Let φ be an LSC submeasure on ω. The following statements are equivalent:*

(1) *φ is a Kadec submeasure;*

(2) *$\varphi \circ \zeta_{\{\alpha\}} \colon (\mathrm{Exh}(\varphi), \tau_\varphi) \to \mathbb{R}$ is continuous for every $\alpha \in \omega$; and*

(3) *$\zeta_{\{\alpha\}} \colon (\mathrm{Exh}(\varphi), \tau_\varphi) \to (\mathrm{Exh}(\varphi), \tau_\varphi)$ is continuous for each $\alpha \in \omega$.*

Proof. First we show that statements (2) and (3) are equivalent, then we show that statements (1) and (3) are equivalent.

(2) $\Leftrightarrow$ (3). Suppose that statement (2) is true. Take $\alpha \in \omega$. Clearly, $\zeta_{\{\alpha\}}$ is continuous with respect to τ_{w}, so it suffices to show that $\zeta_{\{\alpha\}}^{-1}(\varphi^{-1}([0,t)) \cap \mathrm{Exh}(\varphi)) \in \tau_\varphi$ on $\mathrm{Exh}(\varphi)$ for $t \in (0,\infty)$. This follows from the fact that this set equals the set $(\varphi \circ \zeta_{\{\alpha\}})^{-1}([0,t)) \cap \mathrm{Exh}(\varphi)$ which is open with respect to τ_φ on $\mathrm{Exh}(\varphi)$ according to the assumption. The implication (3) $\Rightarrow$ (2) is trivial.

(1) $\Rightarrow$ (3). Suppose that φ is a Kadec submeasure and take $\alpha \in \omega$. Note that for all $X, Y \in \mathrm{Exh}(\varphi)$ we have $d\big(\zeta_{\{\alpha\}}(X), \zeta_{\{\alpha\}}(Y)\big) \leq d(X,Y)$. This means that $\zeta_{\{\alpha\}}\colon (\mathrm{Exh}(\varphi), \tau_d) \to (\mathrm{Exh}(\varphi), \tau_d)$ is continuous. Since $\tau_d = \tau_\varphi$ we have shown that statement (3) holds.

(3) $\Rightarrow$ (1). Suppose that $\zeta_{\{\alpha\}}\colon (\mathrm{Exh}(\varphi), \tau_\varphi) \to (\mathrm{Exh}(\varphi), \tau_\varphi)$ is continuous for every $\alpha \in \omega$. By composing these functions, one can see that $\zeta_F\colon (\mathrm{Exh}(\varphi), \tau_\varphi) \to (\mathrm{Exh}(\varphi), \tau_\varphi)$ is continuous for every finite subset $F \subset \omega$. In view of Lemma 3.3.11 it suffices to show that $\tau_d \subset \tau_\varphi$. So take a set $X \in \mathrm{Exh}(\varphi)$ and an $\varepsilon > 0$ and consider $B_d(X,\varepsilon)$, the ε-ball around X with respect to the metric d as in Theorem 3.2.3. We show that there is an open set O in τ_φ such that $X \in O \subset B_d(X,\varepsilon)$, which suffices to prove that $\tau_d \subset \tau_\varphi$.

As $X \in \mathrm{Exh}(\varphi)$ there is a finite set $F \subset X$ such that $\varphi(X \backslash F) < \varepsilon/2$. Now define

$$O = \{Y \in \mathrm{Exh}(\varphi) : Y \supset F \text{ and } \varphi(Y \setminus F) < \varepsilon/2\}.$$

Clearly, $X \in O$ and

$$\{Y \in \mathrm{Exh}(\varphi) : \varphi(Y \setminus F) < \varepsilon/2\} = \zeta_F^{-1}\Big(\varphi^{-1}\big([0,\varepsilon/2)\big) \cap \mathrm{Exh}(\varphi)\Big),$$

so this set is open in τ_φ on $\mathrm{Exh}(\varphi)$ since $\zeta_F\colon (\mathrm{Exh}(\varphi), \tau_\varphi) \to (\mathrm{Exh}(\varphi), \tau_\varphi)$ is continuous. We see that O is open in τ_φ on $\mathrm{Exh}(\varphi)$. Now take $Y \in O$. We have

$$\begin{aligned} d(X,Y) &\leq \varphi\big((X \triangle Y) \cap F\big) + \varphi\big((X \triangle Y) \setminus F\big) \\ &\leq \varphi\big((X \cap F) \triangle (Y \cap F)\big) + \varphi(X \setminus F) + \varphi(Y \setminus F) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

This means that $O \subset B_d(X,\varepsilon)$. $\square$

The following theorem shows that we can assume without loss of generality that the submeasure φ of Theorem 3.1.2 is a Kadec submeasure and thus our general space $\mathcal{E}$ (with the topology τ_χ) can be reduced to the space $\big(I, \tau_d\big)$ of Theorem 3.1.2.

Theorem 3.3.15. *Let φ be an LSC submeasure on ω. Then there exists a Kadec submeasure ψ on ω such that $\varphi \le \psi \le 2\,\varphi$.*

Proof. Let $\{P_0, P_1, P_2, \ldots\}$ be an enumeration of the finite subsets of ω such that $P_0 = \emptyset$. Now we define $\psi : \mathcal{P}(\omega) \to [0, \infty]$ as follows:

$$\psi(A) = \sum_{n=0}^{\infty} 2^{-n}\,\varphi(A \setminus P_n).$$

Note that since all the terms in the sum are nonnegative ψ is well defined. It follows easily that ψ is a submeasure because we know that φ is one. Furthermore, for all $A \subset \omega$ we have $\psi(A) \ge \varphi(A \setminus P_0) = \varphi(A)$ and $\psi(A) \le \sum_{n=0}^{\infty} 2^{-n}\,\varphi(A) = 2\,\varphi(A)$. We find that

$$\varphi \le \psi \le 2\,\varphi. \tag{3.10}$$

To see that ψ is LSC note that every term in the sum is a nonnegative LSC function on $\mathcal{P}(\omega)$ and that every countable sum of nonnegative LSC functions is again an LSC function.

Let $d_\psi(A, B) = \psi(A \mathbin{\triangle} B)$ for $A, B \in \operatorname{Exh}(\psi)$. It remains to be shown that $\tau_{d_\psi} \subset \tau_\psi$ on $\operatorname{Exh}(\psi)$. Take $X \in \operatorname{Exh}(\psi)$ and let $\varepsilon > 0$. We will show that we can find a set $C \in \tau_\psi$ such that $X \in C \subset B_{d_\psi}(X, \varepsilon)$, where $B_{d_\psi}(X, \varepsilon)$ is the ε-ball with center X and radius ε with respect to the metric d_ψ.

Since $X \in \operatorname{Exh}(\psi)$ we can find a number $k \in \omega$ such that $\psi(X \setminus k) < \varepsilon/4$. Let $N \in \omega$ be such that $P_N = X \cap k$. Then we have $P_N \subset X$ and

$$d_\psi(X, P_N) = \psi(X \mathbin{\triangle} P_N) = \psi(X \setminus P_N) = \psi(X \setminus k) < \varepsilon/4. \tag{3.11}$$

Define $f : 2^\omega \to [0, \infty)$ by

$$f(A) = \psi(A) - 2^{-N}\varphi(A \setminus P_N) = \sum_{n=0}^{N-1} 2^{-n}\varphi(A \setminus P_n) + \sum_{n=N+1}^{\infty} 2^{-n}\varphi(A \setminus P_n).$$

Note that f is LSC with respect to τ_{w}. Next we define $W \in \tau_{\mathrm{w}}$ as

$$W = \{A \subset \omega : A \cap k = P_N\}$$

and we put $\delta = 2^{-N-4}\varepsilon$. We write

$$C = \{Y \in W : \psi(Y) < \psi(X) + \delta \text{ and } f(Y) > f(X) - \delta\}.$$

It is clear that $X \in C$ and that $C \in \tau_\psi$ because f is LSC with respect to τ_{w} and ψ is continuous with respect to τ_ψ. Let $Y \in C$, then $\psi(Y) < \psi(X) + \delta$ and $f(Y) > f(X) - \delta$. From the inequality

$$f(Y) + 2^{-N}\varphi(Y \setminus P_N) < f(X) + 2^{-N}\varphi(X \setminus P_N) + \delta$$

we derive that

$$\begin{aligned} \varphi(Y \setminus P_N) &< \varphi(X \setminus P_N) + 2^N(f(X) - f(Y)) + 2^N\delta \\ &< \psi(X \setminus P_N) + 2^N\delta + 2^N\delta \\ &< 3\,\varepsilon/8. \end{aligned}$$

Since $Y \in W$ we have $Y \setminus P_N = Y \triangle P_N$ and we see that

$$\begin{aligned} d_\psi(Y, X) \le d_\psi(Y, P_N) + d_\psi(P_N, X) &\le 2\varphi(Y \setminus P_N) + \psi(P_N \triangle X) \\ &< \frac{3}{4}\varepsilon + \frac{1}{4}\varepsilon = \varepsilon. \end{aligned}$$

We have shown that $X \in C \subset B_{d_\psi}(X, \varepsilon)$ which proves the theorem. $\square$

Remark 3.3.16. Note that the property $\varphi \le \psi \le 2\,\varphi$ immediately implies that $\mathrm{Exh}(\psi) = \mathrm{Exh}(\varphi)$ and $\mathrm{Fin}(\psi) = \mathrm{Fin}(\varphi)$ and that the metrics d_φ and d_ψ as described in Theorem 3.2.3 on $\mathrm{Exh}(\varphi)$ are uniformly equivalent.

3.4 The generalizing theorem

In this section we will continue in the framework of an LSC function $\chi\colon X \to [0, \infty]$ as introduced in Section 3.3. In particular, χ satisfies properties (a)–(d) from Section 3.3. We will work towards Theorem 3.4.7, which is a generalization of Theorem 3.1.1 and of the equivalence of statements (1)–(3) of Theorem 3.1.2. Remember that $(E_n)_{n\in\omega}$ is a fixed sequence of sets such that $E_n \subset X_n$.

Our first result generalizes the equivalence (1) $\Leftrightarrow$ (2) of Theorem 3.1.1 and Theorem 3.1.2. For $n \in \omega$ and $\varepsilon > 0$ define the subset $A_n(\varepsilon)$ of X_n by

$$A_n(\varepsilon) = \operatorname{Sup}_{R_n}\big(E_n \cap (\chi \circ \alpha_n)^{-1}([0,\varepsilon])\big).$$

By assumption the set $A_n(\varepsilon)$ is not empty (see Section 3.3). Note that all points in $\prod_{n=0}^{\infty} A_n(\varepsilon)$ have the same χ-value. Define the set $B_n(\varepsilon) = \big(E_n \cap (\chi \circ \alpha_n)^{-1}([0,\varepsilon])\big) \cup \{0\}$ for all $n \in \omega$ and observe that $\prod_{n=0}^{\infty} A_n(\varepsilon) = \prod_{n=0}^{\infty} \operatorname{Sup}_{R_n} B_n(\varepsilon)$. It follows from (3.4) that for all $a \in \prod_{n=0}^{\infty} A_n(\varepsilon)$ we have

$$\chi(a) = \sup\{\chi(x) : x \in \prod_{n=0}^{\infty} B_n(\varepsilon)\}. \tag{3.12}$$

For each $\varepsilon > 0$ we pick a point $\eta(\varepsilon) \in \prod_{n=0}^{\infty} A_n(\varepsilon)$.

Remember that $\mathcal{E} = \operatorname{Fin}(\chi) \cap \prod_{n=0}^{\infty} E_n$, so $\mathcal{E} \subset \operatorname{Fin}(\chi)$. To prove Theorem 3.4.7 we need to assume that $\operatorname{Exh}(\chi) = \operatorname{Fin}(\chi)$, however, for the next proposition and Theorem 3.4.2 it suffices to assume that $\mathcal{E} \subset \operatorname{Exh}(\chi)$. This is equivalent to saying that $\operatorname{Exh}(\chi)$ is equal to $\operatorname{Fin}(\chi)$ on $\prod_{n=0}^{\infty} E_n$, i.e.

$$\operatorname{Fin}(\chi) \cap \prod_{n=0}^{\infty} E_n = \operatorname{Exh}(\chi) \cap \prod_{n=0}^{\infty} E_n.$$

Proposition 3.4.1. *Assume that $\mathcal{E}$ is not empty and that $\mathcal{E} \subset \operatorname{Exh}(\chi)$. Then $\chi\big(\eta(\varepsilon)\big) = \infty$ for all $\varepsilon > 0$ if and only if there exists an $x \in \prod_{n=0}^{\infty} E_n$ with $\chi(x) = \infty$ and $\lim_{n\to\infty} \chi_n(x) = 0$ in $\mathbb{R}$.*

Remember that the addition 'in $\mathbb{R}$' means that $\chi_n(x) < \infty$ for all $n \in \omega$.

Proof. For the 'if part' assume that there is an $x \in \prod_{n=0}^{\infty} E_n$ with $\chi(x) = \infty$ and $\lim_{n\to\infty} \chi_n(x) = 0$ with $\chi_n(x) < \infty$ for every $n \in \omega$. Take an $\varepsilon > 0$. We can find an $N \in \omega$ such that $\chi_n(x) \leq \varepsilon$ for all $n \geq N$. Since every $\chi_n(x)$ is finite we have $\chi\big(\zeta_N(x)\big) = \infty$ by Lemma 3.3.9. Note that $\zeta_N(x) \in \prod_{n=0}^{\infty} B_n(\varepsilon)$, which implies that $\zeta_N(x) R\, \eta(\varepsilon)$. Using the monotonicity of χ we conclude that $\chi\big(\eta(\varepsilon)\big) = \infty$.

Assume now that $\chi\big(\eta(\varepsilon)\big) = \infty$ for all $\varepsilon > 0$. First, we shall recursively construct sequences $n_0 < n_1 < \cdots$ in ω and $y_0, y_1, \ldots$ such that for all $k \in \mathbb{N}$,

(i) $y_m \in B_m(1/k)$ for $n_{k-1} \leq m < n_k$ and

(ii) $\chi(y_0, y_1, \ldots, y_{n_k-1}, 0, 0, \ldots) > k$.

Put $n_0 = 0$ and assume that $n_0, \ldots, n_{k-1}$ and $y_0, \ldots, y_{n_{k-1}-1}$ have been found. We have $\chi\big(\eta(1/k)\big) = \infty$ and $\chi_n(\eta(1/k)) \leq 1/k$ for $n \in \omega$, thus $\chi(\zeta_{n_{k-1}}(\eta(1/k))) = \infty$ by Lemma 3.3.9. Using property (d) of χ we can find $b \in \prod_{n=0}^{\infty} B_n(1/k)$ such that $\chi(b) > k$ and $\zeta_{n_{k-1}}(b) = b$. Using (3.6) we can select an $n_k > n_{k-1}$ with the property that $\chi(\xi_{n_k}(b)) > k$. If we define $y_i = b_i$ for $n_{k-1} \leq i < n_k$ then the desired properties are clearly satisfied.

Putting $y = (y_0, y_1, \ldots)$ we see that $\chi(y) = \infty$ by hypothesis (ii) and $\lim_{n\to\infty} \chi_n(y) = 0$ in $\mathbb{R}$ by hypothesis (i). Since $\mathcal{E} \neq \emptyset$ we can select a point $z \in \mathcal{E}$. We define $x \in \prod_{n=0}^{\infty} E_n$ by

$$x_n = \begin{cases} y_n, & \text{if } y_n \neq 0; \\ z_n, & \text{if } y_n = 0. \end{cases}$$

It is clear that yRx so $\chi(y) \leq \chi(x)$ and hence $\chi(x) = \infty$. Furthermore, we have $z \in \mathrm{Exh}(\chi)$, which easily implies that $\lim_{n\to\infty} \chi_n(z) = 0$ in $\mathbb{R}$. We already know that $\lim_{n\to\infty} \chi_n(y) = 0$ in $\mathbb{R}$, so we have $\lim_{n\to\infty} \chi_n(x) = 0$ in $\mathbb{R}$ as well. We see that x is as required. $\square$

Next we prove a generalization of the implication (2) $\Rightarrow$ (3) of Theorem 3.1.1 and Theorem 3.1.2. Just as for submeasures on ω, a subset A of X is called *χ-bounded* if it is bounded with respect to χ, that is, if there is an $M \in \mathbb{N}$ such that $\chi(a) \leq M$ for all $a \in A$; see also §3.1. The set A is called *χ-unbounded* if it is not χ-bounded.

Theorem 3.4.2. *Assume that $\mathcal{E} \subset \mathrm{Exh}(\chi)$ and suppose that there exists a point $x \in \prod_{n=0}^{\infty} E_n$ such that $\chi(x) = \infty$ and $\lim_{n\to\infty} \chi_n(x) = 0$ in $\mathbb{R}$. Then every nonempty clopen subset of $\mathcal{E}$ is χ-unbounded (and hence $\dim \mathcal{E} \neq 0$).*

PROOF. We will show that every nonempty χ-bounded subset A of $\mathcal{E}$ has a boundary point. So suppose $A \subset \mathcal{E}$ is not empty and χ-bounded and let $M \in \mathbb{N}$ be such that $\chi(a) \leq M$ for all $a \in A$. We recursively construct sequences $n_0 < n_1 < \cdots$ in ω, $a^0, a^1, \ldots$ in A, and $b^1, b^2, \ldots$ in $\mathcal{E} \setminus A$ such that for $i \in \mathbb{N}$,

(i) $\xi_{n_{i-1}}(a^i) = \xi_{n_{i-1}}(b^i) = \xi_{n_{i-1}}(a^{i-1})$;

(ii) $|\chi(a^i) - \chi(\xi_{n_i}(a^i))| < 2^{-i}$; and

(iii) $|\chi(b^i) - \chi(\xi_{n_i}(a^i))| < 2^{-i+1}$.

Since A is not empty we can find an $a^0 \in A$ and we put $n_0 = 0$.

Assume now that a^{i-1} and n_{i-1} have been found for some $i \in \mathbb{N}$. Since $a^{i-1} \in A \subset \mathrm{Exh}(\chi)$ and $\lim_{n\to\infty} \chi_n(x) = 0$ we can find a number $k > n_{i-1}$ such that $\chi(\zeta_k(a^{i-1})) < 2^{-i}$ and $\chi_n(x) < 2^{-i}$ for each $n \geq k$. For $j \in \omega$ we define $y^j \in \prod_{i=0}^{\infty} E_i$ by

$$y^j_m = \begin{cases} x_m, & \text{if } k \leq m < k+j; \\ a^{i-1}_m, & \text{if } m < k \text{ or } m \geq k+j. \end{cases}$$

Note that y^j differs from a^{i-1} in only finitely many coordinates and that $\chi_n(x) \in \mathbb{R}$ for all $n \in \omega$, so that $y^j \in \mathcal{E}$ by Lemma 3.3.9. Put $y = (a^{i-1}_0, \ldots, a^{i-1}_{k-1}, x_k, x_{k+1}, \ldots)$ and observe that $\chi(y) = \infty$ by Lemma 3.3.9 again. Note that $\lim_{j\to\infty} y^j = y$ with respect to τ_{w}, thus $\lim_{j\to\infty} \chi(y^j) = \infty$ by the LSC property of χ. Since $y^0 = a^{i-1} \in A$, and A is χ-bounded we can find an $m \in \omega$ such that $a^i = y^m \in A$ and $b^i = y^{m+1} \in \mathcal{E} \setminus A$. Note that hypothesis (i) is clearly satisfied and put $n_i = k + m$. By subadditivity and monotonicity we have

$$\begin{aligned} \chi(\xi_{n_i}(a^i)) \leq \chi(a^i) &\leq \chi(\xi_{n_i}(a^i)) + \chi(\zeta_{n_i}(a^i)) \\ &= \chi(\xi_{n_i}(a^i)) + \chi(\zeta_{n_i}(a^{i-1})) \\ &\leq \chi(\xi_{n_i}(a^i)) + \chi(\zeta_k(a^{i-1})) \\ &< \chi(\xi_{n_i}(a^i)) + 2^{-i}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \chi(\xi_{n_i}(a^i)) \leq \chi(b^i) &\leq \chi(\xi_{n_i}(a^i)) + \chi_{n_i}(x) + \chi(\zeta_{n_i+1}(a^{i-1})) \\ &< \chi(\xi_{n_i}(a^i)) + 2^{-i} + 2^{-i}. \end{aligned}$$

Thus hypotheses (ii) and (iii) are satisfied.

The induction being complete we can use hypothesis (i) to define $c \in \prod_{i=0}^{\infty} E_i$ by $\xi_{n_i}(c) = \xi_{n_i}(a^i)$ for every $i \in \omega$. We have

$$\lim_{i\to\infty} a^i = \lim_{i\to\infty} \xi_{n_i}(a^i) = \lim_{i\to\infty} b^i = c \text{ in } (X, \tau_{\mathrm{w}}). \tag{3.13}$$

Formula (3.6) tells us that $\lim_{i\to\infty} \chi(\xi_{n_i}(a^i)) = \chi(c)$. Since $\chi(\xi_{n_i}(a^i)) \leq \chi(a^i) \leq M$ for all $i \in \omega$ we see that $\chi(c) \leq M$ and hence $c \in \mathcal{E}$. Note that properties (ii) and (iii) also yield that $\lim_{i\to\infty} \chi(a^i) = \lim_{i\to\infty} \chi(b^i) = \chi(c)$. Together with (3.13) we obtain

$$\lim_{i\to\infty} a^i = \lim_{i\to\infty} b^i = c \text{ in } (\mathcal{E}, \tau_\chi).$$

We conclude that c is a boundary point of A in $\mathcal{E}$. $\square$

This theorem shows that it follows from statement (2) of Theorem 3.1.2 that every nonempty clopen subset of $(\mathcal{I}, \tau_d)$ is φ-unbounded. Clearly, this implies that $(\mathcal{I}, \tau_d)$ is a cohesive space, which in turn implies that $\dim(\mathcal{I}, \tau_d) > 0$. We see that we can add the following statements to Theorem 3.1.2:

(i) every nonempty clopen subset of $(\mathcal{I}, \tau_d)$ is φ-unbounded, and

(ii) $(\mathcal{I}, \tau_d)$ is cohesive.

Now we prove a result that is a generalization of (3) $\Rightarrow$ (1) of Theorem 3.1.2 and a partial generalization of (5) $\Rightarrow$ (1) of Theorem 3.1.1. As usual, if p is a point in a space Y then $\mathrm{ind}_p Y$ denotes the dimension of Y at p, see van Mill [24, p. 227]. For instance, $\mathrm{ind}_p Y = 0$ means that p has a clopen neighbourhood basis in Y.

Theorem 3.4.3. *Suppose that* $\mathrm{Fin}(\chi) = \mathrm{Exh}(\chi)$ *and that every set* E_n *is zero-dimensional. Let* $y \in \mathcal{E}$ *be such that* $\chi(y) = 0$ *and suppose that for every* $n \in \omega$ *the function* $\chi \circ \alpha_n \restriction E_n$ *is continuous at the point* y_n*. Then* $\mathrm{ind}_y\, \mathcal{E} > 0$ *implies* $\chi\big(\eta(\varepsilon)\big) = \infty$ *for all* $\varepsilon > 0$.

Proof. We prove this theorem by contraposition. Suppose that there exists an $\varepsilon_0 > 0$ such that $\chi\big(\eta(\varepsilon_0)\big) < \infty$. Take $\varepsilon \in (0, \varepsilon_0]$. Since $\mathcal{C}$ given by formula (3.9) is a basis for the topology on $\mathcal{E}$ and τ_{w} is a zero-dimensional topology on $\prod_{i=0}^{\infty} E_i$, it suffices to find a clopen set U in $\mathcal{E}$ such that $y \in U \subset \chi^{-1}([0,\varepsilon))$. Note that $\eta(\varepsilon_0) \in \mathrm{Fin}(\chi) = \mathrm{Exh}(\chi)$ and hence we can find a number $k \in \mathbb{N}$ such that

$$\chi\big(\zeta_k(\eta(\varepsilon_0))\big) < \varepsilon/2. \tag{3.14}$$

Observe that $\chi_n(y) = \chi(\alpha_n(y_n)) = 0$ for each n. By the continuity of $\chi \circ \alpha_n \restriction E_n$ at y_n we can find a clopen neighbourhood C of y in $\prod_{i=0}^{\infty} E_i$ such that for $x \in C$ and $n < k$ we have

$$\chi_n(x) < \varepsilon/(2k). \tag{3.15}$$

Next define the set U as

$$U = \{x \in C : \chi(x) \leq \varepsilon\}.$$

Note that $y \in U$ and it follows from the lower semi-continuity of χ that U is closed in τ_{w} (and hence in τ_χ). Take an $x \in U$. We have

$$\chi(x) \leq \chi\big(\xi_k(x)\big) + \chi\big(\zeta_k(x)\big), \tag{3.16}$$

and using Lemma 3.3.8 and (3.15) we see that

$$\chi\big(\xi_k(x)\big) \leq \sum_{n=0}^{k-1} \chi_n(x) < \frac{\varepsilon}{2k} k = \varepsilon/2. \tag{3.17}$$

If $n \geq k$ then $\chi_n(x) \leq \chi(x) \leq \varepsilon \leq \varepsilon_0$ and hence $\zeta_k(x) R\, \zeta_k\big(\eta(\varepsilon_0)\big)$ by the definition of $\eta(\varepsilon_0)$. Using the monotonicity of χ and (3.14) we get

$$\chi(\zeta_k(x)) \leq \chi(\zeta_k(\eta(\varepsilon_0))) < \varepsilon/2. \tag{3.18}$$

Combining (3.16), (3.17), and (3.18) it follows that $\chi(x) < \varepsilon$. This means that we can also write $U = C \cap \chi^{-1}([0,\varepsilon))$, which is an open neighbourhood of y in $\mathcal{E}$. We have shown that U is a clopen neighbourhood of y such that $\chi(x) < \varepsilon$ for all $x \in U$. $\square$

Remark 3.4.4. Note that it follows from the lower semi-continuity of χ that $\big(\prod_{n=0}^{\infty} E_n, \tau_{\mathrm{w}}\big)$ is a witness to the almost zero-dimensionality of $\mathcal{E}$ if every set E_n is zero-dimensional.

Combining Proposition 3.4.1 and Theorems 3.4.2 and 3.4.3 we get the following result.

Theorem 3.4.5. *Suppose that* $\mathrm{Fin}(\chi) = \mathrm{Exh}(\chi)$ *and that every set* E_n *is zero-dimensional. Furthermore, suppose there is an* $y \in \mathcal{E}$ *with* $\chi(y) = 0$ *and such that for every* $n \in \omega$ *the function* $\chi \circ \alpha_n \restriction E_n$ *is continuous at* y_n*. Then the following statements are equivalent:*

(1) $\chi\big(\eta(\varepsilon)\big) = \infty$ *for each* $\varepsilon > 0$*;*

(2) *there exists an* $x \in \prod_{n=0}^{\infty} E_n$ *with* $\chi(x) = \infty$ *and* $\lim_{n\to\infty} \chi_n(x) = 0$ *in* $\mathbb{R}$*;*

(3) *every nonempty clopen subset of* $\mathcal{E}$ *is* χ*-unbounded;*

(4) $\mathcal{E}$ *is cohesive;*

(5) $\mathrm{ind}_z\, \mathcal{E} > 0$ *for all* $z \in \mathcal{E}$*; and*

(6) $\mathrm{ind}_y\, \mathcal{E} > 0$.

Motivated by Theorem 3.1.1, we aim at conditions under which statements (1), (2), and (3) of Theorem 3.4.5 are equivalent to the statement that $\dim \mathcal{E} > 0$.

Theorem 3.4.6. *Suppose that* $\mathrm{Fin}(\chi) = \mathrm{Exh}(\chi)$ *and that every set* E_n *is zero-dimensional. Furthermore, suppose that for infinitely many* $m, k \in \mathbb{N}$ *the functions* $\chi \circ \xi_m \restriction (\mathcal{E}, \tau_{\mathrm{w}})$ *and* $\chi \circ \zeta_k \restriction (\mathcal{E}, \tau_\chi)$ *are continuous. If* $\dim \mathcal{E} > 0$ *then* $\chi\big(\eta(\varepsilon)\big) = \infty$ *for all* $\varepsilon > 0$.

Proof. As expected this proof is similar to the proof of Theorem 3.4.3. Suppose that there exists an $\varepsilon_0 > 0$ such that $\chi(\eta(\varepsilon_0)) < \infty$. Take a point $z \in \mathcal{E}$ and let $\varepsilon \in (0, \varepsilon_0]$. It suffices to show that there is a clopen neighbourhood U of z in $\mathcal{E}$ with $U \subset \chi^{-1}([0, \chi(z) + \varepsilon))$.

Since both z and $\eta(\varepsilon_0)$ are elements of $\mathrm{Exh}(\chi)$ we can find $m, k \in \mathbb{N}$ such that $m \geq k$ and

(i) $\chi(\zeta_k(\eta(\varepsilon_0))) < \varepsilon/2$;

(ii) $\chi(\zeta_k(z)) \leq \varepsilon_0$;

(iii) $\chi \circ \xi_m \restriction (\mathcal{E}, \tau_{\mathrm{w}})$ is continuous; and

(iv) $\chi \circ \zeta_k \restriction (\mathcal{E}, \tau_\chi)$ is continuous.

Note that $(\mathcal{E}, \tau_{\mathrm{w}})$ is a subspace of $\prod_{i=0}^\infty E_i$ and therefore zero-dimensional. By (iii) there is a clopen neighbourhood C of z in $(\mathcal{E}, \tau_{\mathrm{w}})$ such that

$$\chi(\xi_m(x)) < \chi(\xi_m(z)) + \varepsilon/2, \tag{3.19}$$

for all $x \in C$. Next define

$$U = \{x \in C : \chi(\zeta_k(x)) \le \varepsilon_0\}$$

and note that $z \in U$ because of (ii). By the lower semi-continuity of χ it follows that U is a closed set in $(\mathcal{E}, \tau_{\mathrm{w}})$ and hence in $(\mathcal{E}, \tau_\chi)$.

Take an $x \in U$. Since $\chi_n(x) \le \chi(\zeta_k(x)) \le \varepsilon_0$ for all $n \ge k$ we have $\zeta_k(x) R\, \zeta_k(\eta(\varepsilon_0))$. With (i) we see that

$$\chi(\zeta_k(x)) \le \chi(\zeta_k(\eta(\varepsilon_0))) < \varepsilon/2 < \varepsilon_0.$$

This means that we can also write $U = \{x \in C : \chi(\zeta_k(x)) < \varepsilon_0\}$, which is an open set in $\mathcal{E}$ with respect to τ_χ because of condition (iv). By the subadditivity of χ and formula (3.19) we have

$$\begin{aligned} \chi(x) \le \chi\big(\xi_k(x)\big) + \chi\big(\zeta_k(x)\big) &< \chi\big(\xi_m(x)\big) + \varepsilon/2 \\ &< \chi(\xi_m(z)) + \varepsilon/2 + \varepsilon/2 \\ &\le \chi(z) + \varepsilon. \end{aligned}$$

We conclude that U is a clopen neighbourhood of z in $\mathcal{E}$ that is contained in the set $\chi^{-1}([0, \chi(z) + \varepsilon))$. $\square$

This result is a generalization of the implication (5) $\Rightarrow$ (1) of Theorem 3.1.1: take χ equal to the p-norm on $\tilde{\mathbb{R}}^\omega$ and let E_n be a zero-dimensional subset of $\mathbb{R}$ for every $n \in \omega$. It is clear that for all $k \in \mathbb{N}$ the function $\chi(\xi_k(x)) = \left(\sum_{i<k} |x_i|^p\right)^{1/p}$ is continuous with respect to the product topology on ℓ^p and that $\|\zeta_k(x)\|_p$ is continuous with respect to the norm topology.

If we assume that χ is a Kadec submeasure on ω, then it follows from Proposition 3.3.14 that $\chi \circ \zeta_k \restriction (\mathcal{E}, \tau_\chi)$ is continuous for all $k \in \mathbb{N}$. Furthermore, $\chi \circ \xi_k \restriction (\mathcal{E}, \tau_w)$ is continuous for all $k \in \omega$ because $\xi_k(\mathcal{E})$ is finite and hence discrete. Thus Theorem 3.4.6 contains the implication (3) $\Rightarrow$ (1) of Theorem 3.1.2 when we also use Theorem 3.3.15 to replace φ with a Kadec submeasure ψ if necessary.

Combining Proposition 3.4.1 and Theorems 3.4.2 and 3.4.6 we get our main result, which is, according to the above observations, a generalization of Theorem 3.1.1 and of the equivalence of the statements (1), (2), and (3) in Theorem 3.1.2.

Theorem 3.4.7. *Suppose that $\mathcal{E}$ is not empty and that* $\mathrm{Fin}(\chi) = \mathrm{Exh}(\chi)$. *Furthermore, assume that every set E_n is zero-dimensional and that for infinitely many $m, k \in \mathbb{N}$ the functions $\chi \circ \xi_m \restriction (\mathcal{E}, \tau_w)$ and $\chi \circ \zeta_k \restriction (\mathcal{E}, \tau_\chi)$ are continuous. Then the following statements are equivalent:*

(1) $\chi\big(\eta(\varepsilon)\big) = \infty$ *for all* $\varepsilon > 0$;

(2) there exists an $x \in \prod_{n=0}^{\infty} E_n$ *with* $\chi(x) = \infty$ *and* $\lim_{n\to\infty} \chi_n(x) = 0$ *in* $\mathbb{R}$;

(3) every nonempty clopen subset of $\mathcal{E}$ is χ-unbounded;

(4) $\mathcal{E}$ is cohesive; and

(5) $\dim \mathcal{E} > 0$.

3.5 The space $\mathcal{E}$ as representation of $\mathfrak{E}_c$

The last statement in Theorem 3.1.2 is that $\big(\mathcal{I}, \tau_d\big)$ is homeomorphic to complete Erdős space. In this section we give conditions under which our general space $\mathcal{E}$ is homeomorphic to $\mathfrak{E}_c$. As a result we get a full generalization of Theorem 3.1.2.

If X is a nonempty space then Y is called an *X-factor* if there is a space Z such that $Y \times Z$ is homeomorphic to X. We need the following results of Dijkstra and van Mill [10, Theorem 12 and 15].

Theorem 3.5.1. *A nonempty space E is an $\mathfrak{E}_c$-factor if and only if E is almost zero-dimensional as witnessed by a topology $\mathcal{W}$ such that every point of E has a neighbourhood that is complete in $(E, \mathcal{W})$.*

Theorem 3.5.2. *A nonempty space is homeomorphic to $\mathfrak{E}_c$ if and only if it is a cohesive $\mathfrak{E}_c$-factor.*

The following result generalizes Theorem 1.2.7, Dijkstra [5, Corollary 4] and Dijkstra and van Mill [10, Theorem 23].

Theorem 3.5.3. *Assume that $\mathcal{E}$ is not empty and that the sets E_n are zero-dimensional and topologically complete. Then $\mathcal{E}$ is an $\mathfrak{E}_c$-factor. If, in addition, $\mathcal{E} \subset \mathrm{Exh}(\chi)$ and there is an $x \in \prod_{n=0}^{\infty} E_n$ such that $\chi(x) = \infty$ and $\lim_{n\to\infty} \chi_n(x) = 0$ in $\mathbb{R}$, then $\mathcal{E}$ is homeomorphic to $\mathfrak{E}_c$.*

Proof. We observed in Remark 3.4.4 that $\mathcal{E}$ is almost zero-dimensional as is witnessed by the topology τ_w it inherits from the zero-dimensional and complete space $\prod_{n=0}^{\infty} E_n$. Let $x \in \mathcal{E}$ and consider $U = \chi^{-1}([0, \chi(x)+1]) \cap \prod_{n=0}^{\infty} E_n$. Note that $U \subset \mathrm{Fin}(\chi)$ so U is a neighbourhood of x in $\mathcal{E}$. Since χ is LSC on X we have that U is a closed subspace of the complete space $\prod_{n=0}^{\infty} E_n$ and hence U is topologically complete in the topology τ_w. With Theorem 3.5.1 we see that $\mathcal{E}$ is an $\mathfrak{E}_c$-factor. The second part follows immediately from Theorems 3.4.2 and 3.5.2. $\square$

Corollary 3.5.4. *For every LSC submeasure φ on ω the space $(\mathrm{Fin}(\varphi), \tau_\varphi)$ is an $\mathfrak{E}_c$-factor.*

Corollary 3.5.5. *Suppose that $\mathcal{E}$ is a nonempty subset of $\mathrm{Exh}(\chi)$ and that there is an $x \in \prod_{n=0}^{\infty} E_n$ such that $\chi(x) = \infty$ and $\lim_{n\to\infty} \chi_n(x) = 0$ in $\mathbb{R}$. Then $\mathcal{E}$ contains a closed copy of $\mathfrak{E}_c$.*

Proof. Let $y \in \mathcal{E}$ and define $E'_n = \{x_n, y_n\}$ for $n \in \omega$. Let the space $\mathcal{E}'$ be given by $\mathcal{E}' = \mathrm{Fin}(\chi) \cap \prod_{i=0}^{\infty} E'_i$ with the topology τ_χ. It is clear that $\mathcal{E}'$ is a closed subspace of $\mathcal{E}$ and that Theorem 3.5.3 shows that $\mathcal{E}'$ is homeomorphic to $\mathfrak{E}_c$. $\square$

Remark 3.5.6. It was observed by Dijkstra and van Mill [10, Theorem 23] that if $\mathcal{E}$ is a nonempty subspace of ℓ^p as in Theorem 3.1.1

then $\mathcal{E}$ is an $\mathfrak{E}_c$-factor if and only if every E_n is topologically complete and zero-dimensional. The converse of Theorem 3.5.3 is no longer valid in the general setting as the following examples show. For both examples we start with a space $\mathcal{E}$ that is homeomorphic to $\mathfrak{E}_c$ and we add an extra factor X_{-1} to the product space X of §3.3.

Let $E_{-1} = X_{-1} = \mathfrak{E}_c$ and put $X' = X_{-1} \times X$. We define $\chi' \colon X' \to [0,\infty]$ by $\chi'(x_{-1}, x_0, \dots) = \chi(x_0, x_1, \dots)$ and we let R_{-1} be the full relation $X_{-1} \times X_{-1}$. Then, clearly, the corresponding space $\mathcal{E}'$ equals $\mathfrak{E}_c \times \mathcal{E}$ so it is homeomorphic to $\mathfrak{E}_c$. In this case one of the E_n's is one-dimensional. Note that we cannot weaken the premise of Theorem 3.5.3 to the requirement that the E_n's are almost zero-dimensional because we can also choose $E_{-1} = \mathfrak{E}_c^\omega$. In that case the resulting $\mathcal{E}'$ is homeomorphic to $\mathfrak{E}_c^\omega$, which is not an $\mathfrak{E}_c$-factor according to Dijkstra, van Mill and Steprāns [12].

Now let $E_{-1} = \mathbb{Q}$ and $X_{-1} = \mathbb{Q} \cup \{\infty\} \subset \tilde{\mathbb{R}}$. Let $q_0, q_1, \dots$ be a one-to-one enumeration of $\mathbb{Q}$ and put $q_\infty = \infty$. Define the LSC function $\psi \colon X_{-1} \to [0,\infty]$ by $\psi(q_n) = n$ for $n \in \omega \cup \{\infty\}$ and the order R_{-1} on X_{-1} by $pR_{-1}q$ if $\psi(p) \le \psi(q)$. Now let $\chi'(x) = \psi(x_{-1}) + \chi(x_0, x_1, \dots)$ be defined on $X' = X_{-1} \times X$ and consider the resulting space $\mathcal{E}'$. Note that ψ generates the discrete topology on E_{-1} and hence $\mathcal{E}'$ is the product of a countable discrete space with a complete Erdős space so $\mathcal{E}' \approx \mathfrak{E}_c$. Note that E_{-1} is not topologically complete.

The following result, when combined with Theorem 3.4.7, fully generalizes Theorem 3.1.2.

Theorem 3.5.7. *Suppose that* $\mathrm{Fin}(\chi) = \mathrm{Exh}(\chi)$ *and that every* E_n *is a zero-dimensional and topologically complete space. Furthermore, assume that for infinitely many* $m, k \in \mathbb{N}$ *the functions* $\chi \circ \xi_m \restriction (\mathcal{E}, \tau_w)$ *and* $\chi \circ \zeta_k \restriction (\mathcal{E}, \tau_\chi)$ *are continuous. Then* $\mathcal{E}$ *is homeomorphic to* $\mathfrak{E}_c$ *if and only if* $\dim \mathcal{E} > 0$.

Proof. Clearly, if $\mathcal{E}$ is homeomorphic to $\mathfrak{E}_c$, then $\dim \mathcal{E} > 0$. Now suppose that $\dim \mathcal{E} > 0$, so in particular $\mathcal{E} \neq \emptyset$. It follows from Theorem 3.4.7 that there is an $x \in \prod_{n=0}^\infty E_n$ such that $\chi(x) = \infty$ and $\lim_{n\to\infty} \chi_n(x) = 0$ in $\mathbb{R}$. With Theorem 3.5.3 we conclude that $\mathcal{E} \approx \mathfrak{E}_c$. $\square$

3.6 The space $\mathcal{E}$ as representation of $\mathfrak{E}$

In this section we give a generalization of the following result of Dijkstra and van Mill [11, Proposition 8.26].

Proposition 3.6.1. *Let χ be the p-norm on $\tilde{\mathbb{R}}^\omega$ for some $p \geq 1$ and let $E_n \subset \mathbb{R}$ be a zero-dimensional $F_{\sigma\delta}$-space for all $n \in \omega$ such that* $\dim \mathcal{E} > 0$. *If infinitely many of the E_n's are of the first category in themselves, then $\mathcal{E}$ is homeomorphic to $\mathfrak{E}$.*

Remember the definitions of a tree and an anchor, see Definition 1.3.2 and Definition 1.3.3. To formulate the next theorem we introduce the concept of a *Sierpiński stratification*.

Definition 3.6.2. Let X be a space. We call a system $(X_s)_{s\in T}$ a *Sierpiński stratification* of X if T is a nonempty tree over a countable alphabet and X_s is a closed subset of X for each $s \in T$ such that

(i) $X_\lambda = X$ and $X_s = \bigcup\{X_t : t \in \mathrm{succ}(s)\}$ for all $s \in T$ and

(ii) if $\sigma \in [T]$ then the sequence $X_{\sigma\restriction 0}, X_{\sigma\restriction 1}, \ldots$ converges to a point $x_\sigma \in X$.

Sierpiński [27] proved the following theorem.

Theorem 3.6.3. *A space X is an* (*absolute*) *$F_{\sigma\delta}$-space if and only if it admits a Sierpiński stratification.*

Note that if $(X_s)_{s\in T}$ is a Sierpiński stratification of X, then the whole space X is an anchor, see Definition 1.3.3.

Remark 3.6.4. Let Y be an $F_{\sigma\delta}$-space that is a witness to the almost zero-dimensionality of a space X. Thus X is a subset of Y and we let Z be the set X with the topology that is inherited from Y. Let $(Y_s)_{s\in T}$ be a Sierpiński stratification of Y and put $Z_s = Y_s \cap Z$ for $s \in T$. Let $x \in X$ and choose a neighbourhood B of x in X such that B is closed in Y. If $\sigma \in [T]$ is such that $Y_{\sigma\restriction k} \cap B \neq \emptyset$ for each $k \in \omega$ then $Y_{\sigma\restriction 0}, Y_{\sigma\restriction 1}, \ldots$ converges in Y to a point that must lie in B. Hence $Z_{\sigma\restriction 0}, Z_{\sigma\restriction 1}, \ldots$ converges in Z and B is an anchor for $(Z_s)_{s\in T}$ in Z.

For convenience we state Theorem 1.3.5 again from DIJKSTRA and VAN MILL [11].

Theorem 3.6.5. *A nonemtpy space E is homeomorphic to $\mathfrak{E}$ if and only if there exists a topology $\mathcal{T}$ on E that witnesses the almost zero-dimensionality of E and there exist a nonempty tree T over a countable set and subspaces E_s of E that are closed with respect to $\mathcal{T}$ for each $s \in T$ such that:*

(1) $E_\lambda = E$ and $E_s = \bigcup\{E_t : t \in \operatorname{succ}(s)\}$ whenever $s \in T$;

(2) each $x \in E$ has a neighbourhood U that is an anchor for $(E_s)_{s\in T}$ in $\big(E, \mathcal{T}\big)$;

(3) for each $s \in T$ and $t \in \operatorname{succ}(s)$ the set E_t is nowhere dense in E_s; and

(4) E is $\{E_s : s \in T\}$-cohesive.

In §1.3 we showed that $\mathfrak{E}$ indeed satisfies the conditions of Theorem 3.6.5. We can now prove the following generalization of Proposition 3.6.1.

Theorem 3.6.6. *Suppose that $\mathcal{E}$ is a nonempty subset of* $\operatorname{Exh}(\chi)$ *such that the function $\chi \circ \xi_k{\restriction}(\mathcal{E}, \tau_{\mathrm{w}})$ is continuous for infinitely many $k \in \mathbb{N}$ and suppose that every E_n is a zero-dimensional $F_{\sigma\delta}$-space. Assume that there is an $x \in \prod_{n=0}^\infty E_n$ such that $\chi(x) = \infty$ and* $\lim_{n\to\infty} \chi_n(x) = 0$ *in* $\mathbb{R}$. *If infinitely many of the E_n's are of the first category in themselves then $\mathcal{E}$ is homeomorphic to $\mathfrak{E}$.*

PROOF. Let $\nu\colon \omega \to \omega$ be a bijection such that $E_{\nu(2n)}$ is of the first category in itself for each $n \in \omega$. We use Theorem 3.6.5 to show that $\mathcal{E}$ is homeomorphic to $\mathfrak{E}$. Let W be the zero-dimensional product space $\prod_{n=0}^\infty E_n \subset \prod_{n=0}^\infty X_n$. Since χ is an LSC function on W we see that W witnesses the almost zero-dimensionality of $\mathcal{E}$. Let $\mathcal{T}$ be the witness topology on $\mathcal{E}$ that is inherited from W. Because E_n is an $F_{\sigma\delta}$-space we may choose a Sierpiński stratification $(Z^n_s)_{s\in T_n}$ for E_n such that Z^n_s is not empty for every $s \in T_n$. Since $E_{\nu(2n)}$ is of the first category in itself we may assume that $Z_t^{\nu(2n)}$ is nowhere dense in $Z_\lambda^{\nu(2n)} = E_{\nu(2n)}$ for every $t \in T_{\nu(2n)}$ with $|t| = 1$. We now construct a tree $\mathfrak{T}$ as follows:

$$\mathfrak{T} = \{(s_0, \ldots, s_k, s'_0, \ldots, s'_k) : s_i \in T_{\nu(2i)}, s'_i \in T_{\nu(2i+1)}, \\ \text{and } |s_i| = |s'_i| = k - i \text{ for } 0 \le i \le k \text{ where } k \in \omega\}.$$

If $s = (s_0, \ldots, s_k, s'_0, \ldots, s'_k) \in \mathfrak{T}$ and $t = (t_0, \ldots, t_l, t'_0, \ldots, t'_l) \in \mathfrak{T}$ then $s \prec t$ means that $k \le l, s_i \prec t_i$ and $s'_i \prec t'_i$ for every $i \le k$. Now $\mathfrak{T}$ does not formally satisfy the definition of a tree, but it is obviously isomorphic to a countable tree. Note that in this interpretation we have $|(s_0, \ldots, s_k, s'_0, \ldots, s'_k)| = k$. Let $s = (s_0, \ldots, s_k, s'_0, \ldots, s'_k) \in \mathfrak{T}$ and define the following closed subset of W:

$$W_s = \{w \in W : w_{\nu(2i)} \in Z_{s_i}^{\nu(2i)} \text{ and } w_{\nu(2i+1)} \in Z_{s'_i}^{\nu(2i+1)} \text{ for } i \le k\}.$$

Let $\mathcal{E}_s \subset \mathcal{E}$ stand for $\mathrm{Fin}(\chi) \cap W_s$ with the topology τ_χ. We prove the following claim.

Claim 3.6.7. *Let $\hat{E}_n$ be a subset of X_n for every $n \in \omega$ and put $\hat{\mathcal{E}} = \mathrm{Fin}(\chi) \cap \prod_{n=0}^\infty \hat{E}_n$. If $\hat{E}_n = E_n$ for all but finitely many n, then every nonempty clopen subset of $\hat{\mathcal{E}}$ is χ-unbounded.*

PROOF. If $\hat{\mathcal{E}}$ is empty then the claim is trivially true. So let $y \in \hat{\mathcal{E}}$. Define $z \in \prod_{n=0}^\infty \hat{E}_n$ as

$$z_n = \begin{cases} y_n, & \text{if } \hat{E}_n \ne E_n; \\ x_n, & \text{if } \hat{E}_n = E_n. \end{cases}$$

Then $\chi(z) = \infty$ by Lemma 3.3.9 and also $\lim_{n\to\infty} \chi_n(z) = 0$ in $\mathbb{R}$. Now apply Theorem 3.4.2 to $\hat{\mathcal{E}}$. $\diamondsuit$

With this claim we see that every nonempty clopen subset of $\mathcal{E}_s$ is χ-unbounded. This means that $\mathcal{E}$ is $\{\mathcal{E}_s : s \in \mathfrak{T}\}$-cohesive: just choose χ-bounded neighbourhoods for the points of $\mathcal{E}$. We see that condition (4) of Theorem 3.6.5 is satisfied.

It is easily verified that $(W_s)_{s\in\mathfrak{T}}$ is a Sierpiński stratification of W because it is a product of Sierpiński stratifications. This also means that $(\mathcal{E}_s)_{s\in\mathfrak{T}}$ satisfies condition (1) of Theorem 3.6.5. Moreover, since

W is a witness, condition (2) is easily seen to be satisfied as well, see Remark 3.6.4. We now verify condition (3) of Theorem 3.6.5. Let $s = (s_0, \ldots, s_k, s'_0, \ldots, s'_k) \in \mathfrak{T}$ and let $t = (t_0, \ldots, t_{k+1}, t'_0, \ldots, t'_{k+1}) \in \operatorname{succ}(s)$. Take $y \in \mathcal{E}_t$ and let $O \cap \chi^{-1}([0, \chi(y) + \varepsilon))$ be a basic neighbourhood of y in $\mathcal{E}$, where O is an element of τ_{w} and $\varepsilon > 0$. Select an $m > \nu(2k)$ such that $\{y_0\} \times \cdots \times \{y_{m-1}\} \times X_m \times X_{m+1} \times \cdots \subset O$, $\chi(\zeta_m(y)) < \varepsilon/2$ and $\chi \circ \xi_m \restriction (\mathcal{E}, \tau_{\mathrm{w}})$ is continuous. Then we can find an open set $U \subset E_{\nu(2k)}$ with $y_{\nu(2k)} \in U$ such that

$$\{y_0\} \times \cdots \times \{y_{\nu(2k)-1}\} \times U \times \{y_{\nu(2k)+1}\} \times \cdots \times \{y_{m-1}\} \times \{0\} \times \{0\} \times \cdots \\ \subset O \cap \chi^{-1}([0, \chi(y) + \varepsilon/2)). \quad (3.20)$$

Because $y \in \mathcal{E}_t$, we have $y_{\nu(2k)} \in Z^{\nu(2k)}_{t_k}$ with $|t_k| = (k+1) - k = 1$, so $Z^{\nu(2k)}_{t_k}$ is nowhere dense in $E_{\nu(2k)}$. This means that we can pick a point $q \in U \setminus Z^{\nu(2k)}_{t_k}$. We define $z \in \mathcal{E} \setminus \mathcal{E}_t$ by

$$z_i = \begin{cases} q, & \text{if } i = \nu(2k); \\ y_i, & \text{if } i \neq \nu(2k). \end{cases}$$

Since $y_{\nu(2i)} \in Z^{\nu(2i)}_{t_i} \subset Z^{\nu(2i)}_{s_i}$ for $i < k$, $q \in E_{\nu(2k)} = Z^{\nu(2k)}_{\lambda} = Z^{\nu(2k)}_{s_k}$, and $y_{\nu(2i+1)} \in Z^{\nu(2i+1)}_{t'_i} \subset Z^{\nu(2i+1)}_{s'_i}$ for $i \leq k$ we have $z \in \mathcal{E}_s$. Furthermore, with (3.20) we see that

$$\chi(z) \leq \chi(\xi_m(z)) + \chi(\zeta_m(y)) < \chi(y) + \varepsilon/2 + \varepsilon/2$$

thus $z \in O \cap \chi^{-1}([0, \chi(y) + \varepsilon))$. This shows that $\mathcal{E}_t$ is nowhere dense in $\mathcal{E}_s$, which means that all premises of Theorem 3.6.5 have been verified. □

3.7 A fixed point property

Let Ω be a point not in X, and define the set $\mathcal{E}^+$ as $\mathcal{E}^+ = \mathcal{E} \cup \{\Omega\}$. We suppose that we have a (separable, metric) topology $\tau_{\mathcal{E}^+}$ on $\mathcal{E}^+$ such that $\mathcal{E}$ is a subspace of $\mathcal{E}^+$ and such that $\mathcal{E} \setminus U$ is χ-bounded for every neighbourhood U of Ω. The following definition is taken from Abry, Dijkstra and van Mill [2].

Definition 3.7.1. Let X be a space and $p \in X$. We say that p is a *fixed point* of X if for every nonconstant continuous function $f\colon X \to X$ we have $f(p) = p$.

It is clear that if a space contains a fixed point then it has the fixed point property. The converse is not true: every non-degenerate compact AR is an example of a space with the fixed point property but without a fixed point.

We want to show that under certain conditions the point Ω is a fixed point of $\mathcal{E}^+$. As a result we get a generalization of the following theorem, proved by Abry, Dijkstra and van Mill [2, Theorem 16].

Theorem 3.7.2. *Take $p \geq 1$ and let χ be the p-norm on $\tilde{\mathbb{R}}^\omega$. Let E_n be a zero-dimensional subset of $\mathbb{R}$ for every $n \in \omega$. Then the following statements about $\mathcal{E}^+$ are equivalent:*

(1) Ω is a fixed point of $\mathcal{E}^+$;

(2) $\mathcal{E}^+$ has the fixed point property;

(3) $\mathcal{E}^+$ is connected; and

(4) $\dim \mathcal{E} \neq 0$.

Definition 3.7.3. A space X is called *hereditarily disconnected* if each component of X consists of a single element.

For clarity: a space X is called *totally disconnected* if for every two distinct points $x, y \in X$ we can find a clopen subset C of X such that $x \in C$ and $y \notin C$. Clearly, every totally disconnected space is hereditarily disconnected. However, the converse is not true, see Engelking [19, Problem 6.3.23].

To prove the next theorem we need the following lemma, which can be found in Abry, Dijkstra and van Mill [2, Lemma 14].

Lemma 3.7.4. *Let p be a point in a space X such that $X \setminus \{p\}$ is hereditarily disconnected. If for every open neighbourhood U of p with $U \neq X$ the component of p in U is not closed in X, then p is a fixed point of X.*

Theorem 3.7.5. *Assume that $\mathcal{E} \subset \mathrm{Exh}(\chi)$ and that there exists an $x \in \prod_{n=0}^{\infty} E_n$ such that $\chi(x) = \infty$ and $\lim_{n\to\infty} \chi_n(x) = 0$ in $\mathbb{R}$. Furthermore, suppose that $\mathcal{E}$ is hereditarily disconnected. Then Ω is a fixed point of $\mathcal{E}^+$.*

Proof. In view of Lemma Lemma 3.7.4, let U be an open neighbourhood of Ω in $\mathcal{E}^+$ such that $A = \mathcal{E} \setminus U \neq \emptyset$. Let C be the component of Ω in U. Since U is a neighbourhood of Ω we know that A is χ-bounded. Let $N \in \mathbb{N}$ be such that $A \subset \chi^{-1}([0, N])$. If $y \in \mathcal{E}$ and $k \in \mathbb{N}$ we define $Y_k(y) = \{z \in \mathcal{E} : \xi_k(z) = \xi_k(y)\}$. Changing finitely many coordinates x_n of x into y_n does not affect the properties $\chi(x) = \infty$ (see Lemma 3.3.9) and $\lim_{n\to\infty} \chi_n(x) = 0$. With Theorem 3.4.2 we get that every nonempty clopen subset of each $Y_k(y)$ is χ-unbounded.

Now we recursively construct a sequence $a^0, a^1, \ldots$ of points in A and numbers $n_0 < n_1 < \cdots$ in $\mathbb{N}$ such that for all $i \in \mathbb{N}$

(i) $a^i \in Y_{n_{i-1}}(a^{i-1})$ and

(ii) $\chi\big(\xi_{n_i}(a^i)\big) > s_i - 2^{-i}$,

where $s_i = \sup\big\{\chi(x) : x \in A \cap Y_{n_{i-1}}(a^{i-1})\big\}$.

We start with choosing a point $a^0 \in A$ and we take $n_0 = 1$. Observe that properties (i) and (ii) do not apply to this case. Now assume that a^i and n_i have been found for some $i \in \omega$. Since $s_{i+1} \leq N$ we can choose $a^{i+1} \in A \cap Y_{n_i}(a^i)$ such that

$$\chi(a^{i+1}) > \sup\big\{\chi(x) : x \in A \cap Y_{n_i}(a^i)\big\} - 2^{-i-1} = s_{i+1} - 2^{-i-1}.$$

With formula (3.6) it follows that we can find $n_{i+1} > n_i$ such that $\chi\big(\xi_{n_{i+1}}(a^{i+1})\big) > s_{i+1} - 2^{-i-1}$. Properties (i) and (ii) are satisfied.

By property (i) we can define $c \in \prod_{n=0}^{\infty} E_n$ such that

$$\xi_{n_i}(c) = \xi_{n_i}(a^i),$$

for all $i \in \omega$. Clearly, the sequence $(a^i)_{i\in\omega}$ converges to c in $\prod_{n=0}^{\infty} E_n$. Furthermore, with (3.6) and the monotonicity of χ we get

$$\chi(c) = \lim_{i\to\infty} \chi\big(\xi_{n_i}(c)\big) = \lim_{i\to\infty} \chi\big(\xi_{n_i}(a^i)\big) \leq \sup_{i\in\omega} \chi(a^i) \leq N, \tag{3.21}$$

so $c \in \mathcal{E}$. Note that by properties (i) and (ii) and the monotonicity of χ we have $\chi(a^i) \geq \chi\big(\xi_{n_i}(a^i)\big) > s_i - 2^{-i} \geq \chi(a^i) - 2^{-i}$ for all $i \in \mathbb{N}$. From (3.21) we know that $\lim_{i\to\infty} \chi\big(\xi_{n_i}(a^i)\big) = \chi(c)$, so we get $\lim_{i\to\infty} \chi(a^i) = \chi(c)$. We now have that $c = \lim_{i\to\infty} a^i$ in $\mathcal{E}$ and since A is closed we see that $c \in A$.

For each $i \in \mathbb{N}$ we choose $b^i \in Y_{n_{i-1}}(a^{i-1})$ such that $\chi(b^i) = \chi(a^i) + 2^{-i}$. This is possible because if such a point b^i does not exist, then $\{x \in Y_{n_{i-1}}(a^{i-1}) : \chi(x) < \chi(a^i) + 2^{-i}\}$ is a χ-bounded clopen subset of $Y_{n_{i-1}}(a^{i-1})$ that contains a^i. Clearly, the sequence $(b^i)_i$ converges to c in τ_{w} and $\lim_{i\to\infty} \chi(b^i) = \chi(c)$, so $\lim_{i\to\infty} b^i = c$ in $\mathcal{E}$. With property (ii) we see that $\chi(b^i) \geq \chi(\xi_{n_i}(a^i)) + 2^{-i} > s_i$, hence there is a number $k > n_{i-1}$ such that $\chi(\xi_k(b^i)) > s_i$. Note that $Y_k(b_i) \subset Y_{n_{i-1}}(a^{i-1}) \setminus A \subset U$. If K is a clopen subset of $B = Y_k(b_i) \cup \{\mathit{\Omega}\}$ that does not contain $\mathit{\Omega}$ then K is a χ-bounded and clopen subset of $Y_k(b^i)$ and hence empty. Thus B is a connected subset of U and hence $b^i \in C$ for each i. This means that c is a point in the closure of C that is not in U. We conclude that C is not closed in $\mathcal{E}$ and we can apply Lemma 3.7.4 to obtain that $\mathit{\Omega}$ is a fixed point of $\mathcal{E}^+$. $\square$

Combining Theorem 3.4.7 and Theorem 3.7.5 we get the following corollary which is a generalization of Theorem 3.7.2.

Corollary 3.7.6. *Suppose that* $\mathrm{Fin}(\chi) = \mathrm{Exh}(\chi)$ *and that every set* E_n *is zero-dimensional. Furthermore, assume that for infinitely many* $m, k \in \mathbb{N}$ *the functions* $\chi \circ \xi_m \restriction (\mathcal{E}, \tau_{\mathrm{w}})$ *and* $\chi \circ \zeta_k \restriction (\mathcal{E}, \tau_\chi)$ *are continuous. Then the following statements are equivalent:*

(1) $\mathit{\Omega}$ *is a fixed point of* $\mathcal{E}^+$*;*

(2) $\mathcal{E}^+$ *has the fixed point property;*

(3) $\mathcal{E}^+$ *is connected; and*

(4) $\dim \mathcal{E} \neq 0$.

Proof. The implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are obvious and (3) $\Rightarrow$ (4) follows from Engelking [18, Corollary 1.5.6]. The implication (4) $\Rightarrow$ (1) follows from Theorem 3.4.7 and 3.7.5, where we note that $\mathcal{E}$ is totally disconnected because the sets E_n are zero-dimensional. $\square$

Chapter 4

Nonseparable complete Erdős spaces and submeasures on uncountable cardinals

4.1 Introduction

In the previous chapter we generalized Theorem 3.1.1, dealing with the Erdős type subspaces $\mathcal{E}$ of ℓ^p, by replacing the norm function $\|\cdot\|_p$ on $\mathbb{R}^\omega$ by a more general LSC function on a more general product space. Dijkstra and van Mill [10] used Theorem 3.1.1 together with topological characterizations of $\mathfrak{E}_c$ from [10], to determine exactly when these Erdős type subspaces of ℓ^p are homeomorphic to $\mathfrak{E}_c$. Dijkstra, van Mill and Valkenburg [13, Theorem 1] generalize Theorem 3.1.1 in another way, namely by extending the norm function to an uncountable product $\mathbb{R}^\kappa$ of $\mathbb{R}$ with corresponding Erdős type spaces $\mathcal{E}_\kappa$. In analogy to the separable metric case, they used this generalization and the topological characterizations of $\mathfrak{E}_c$ in [10], to characterize when the spaces $\mathcal{E}_\kappa$ are homeomorphic to a so called *nonseparable complete Erdős space*, see Definition 4.2.6 and the remarks thereafter. Inspired by these results we now want to extend Theorem 3.1.2 of Chapter 3 to the case of submeasures on uncountable cardinal numbers. In view of property (4) of this theorem and the mentioned results we are particularly interested in the question when the related exhaustive ideal is homeomorphic to a nonseparable complete Erdős space.

Dijkstra and van Mill use their characterization of complete Erdős space, [10, Theorem 1], to prove Theorem 3.1.2. In fact, this theorem is a simplified version of [10, Theorem 37]. The next result quotes this theorem with item (6) as an additional equivalent statement which follows from Theorem 3.4.7. In correspondence with Chapter 3 we use the following terminology. Let $f\colon Z \to [0,\infty]$ be an arbitrary function. A subset C of Z is called *f-bounded* if there is an $M \in \mathbb{N}$ such that $f(z) \le M$ for all $z \in C$ and *f-unbounded* otherwise. In the previous chapters we also used the convention that if f is a norm $|\cdot|$ on a Banach space Z, then we simply call a subset C of Z bounded if it is $|\cdot|$-bounded, and unbounded if it is $|\cdot|$-unbounded.

Theorem 4.1.1. *Let φ be an LSC submeasure on ω with $\mathrm{Exh}(\varphi) = \mathrm{Fin}(\varphi)$. Let τ_d be the topology on $\mathrm{Exh}(\varphi)$ that is generated by the metric $d(X,Y) = \varphi(X \triangle Y)$ for $X, Y \in \mathrm{Exh}(\varphi)$ and write $\mathcal{I}_\omega = (\mathrm{Exh}(\varphi), \tau_d)$. Then the following statements are equivalent:*

(1) *$\mathcal{I}_\omega$ is not homeomorphic to $\mathbb{Z}$, 2^ω, or $\mathbb{Z} \times 2^\omega$;*

(2) *there is no $B \subset \omega$ with $\mathrm{Exh}(\varphi) = \{X \subset \omega : X \cap B$ is finite$\}$;*

(3) *for every $\varepsilon > 0$ we have $\varphi(\{n \in \omega : \varphi(\{n\}) \le \varepsilon\}) = \infty$;*

(4) *there is a $B \subset \omega$ with $\varphi(B) = \infty$ and $\lim_{n\to\infty} \varphi(\{n\} \cap B) = 0$;*

(5) *$\mathcal{I}_\omega$ is homeomorphic to $\mathfrak{E}_\mathrm{c}$;*

(6) *every nonempty clopen subset of $\mathcal{I}_\omega$ is φ-unbounded;*

(7) $\operatorname{ind} \mathcal{I}_\omega > 0$*; and*

(8) *$\mathcal{I}_\omega$ is not locally compact.*

We write $\mathcal{I}_\omega$ instead of $\mathcal{I}$ as in Theorem 3.1.2 because we want to study submeasures on arbitrary cardinals and to avoid confusion we add the corresponding cardinal as a subscript of $\mathcal{I}$. The following theorem is an extension of the previous one to LSC submeasures on arbitrary infinite cardinal numbers. We write $\lim_{\alpha\in\mu} x_\alpha = 0$ for a set μ with $|\mu| \ge \omega$ and real numbers x_α for every $\alpha \in \mu$, if for each $\varepsilon > 0$ the set $\{x_\alpha : |x_\alpha| \ge \varepsilon\}$ is finite. If κ is a cardinal number we denote by κ_D the cardinal κ equipped with the discrete topology.

Theorem 4.1.2. *Let φ be an LSC submeasure on some cardinal number $\kappa > \omega$ with $\mathrm{Exh}(\varphi) = \mathrm{Fin}(\varphi)$. Let τ_d be the topology on $\mathrm{Exh}(\varphi)$ generated by the metric $d(X,Y) = \varphi(X \triangle Y)$ for $X, Y \in \mathrm{Exh}(\varphi)$ and write $\mathcal{I}_\kappa = (\mathrm{Exh}(\varphi), \tau_d)$. Then the following statements are equivalent:*

(1) *$\mathcal{I}_\kappa$ is not homeomorphic to κ_{D} or $\kappa_{\mathrm{D}} \times 2^\omega$;*

(2) *there is no $B \subset \kappa$ with $\mathrm{Exh}(\varphi) = \{X \subset \kappa : X \cap B$ is finite$\}$;*

(3) *for every $\varepsilon > 0$ we have $\varphi(\{\alpha \in \kappa : \varphi(\{\alpha\}) \leq \varepsilon\}) = \infty$;*

(4) *there is a $B \subset \kappa$ with $\varphi(B) = \infty$ and $\lim_{\alpha \in B} \varphi(\{\alpha\}) = 0$;*

(5) *$\mathcal{I}_\kappa \times \mathfrak{E}_{\mathrm{c}}$ is homeomorphic to $\mathcal{I}_\kappa$;*

(6) *every nonempty clopen subset of $\mathcal{I}_\kappa$ is φ-unbounded;*

(7) *$\mathrm{ind}\, \mathcal{I}_\kappa > 0$; and*

(8) *$\mathcal{I}_\kappa$ is not locally compact.*

We prove this theorem in §4.4. Note that statement (5) above is weaker than its counterpart in Theorem 4.1.1. This is due to the non-separability of the considered spaces $\mathcal{I}_\kappa$, whilst $\mathfrak{E}_{\mathrm{c}}$ is separable. However, in §4.5 we show that for a special class of submeasures we have a stronger analogy with Theorem 4.1.1. There we prove that for these special submeasures we can replace statement (5) of Theorem 4.1.2 by the statement that $\mathcal{I}_\kappa$ is homeomorphic to $\mathfrak{E}_{\mathrm{c}} \times (\lambda_{\mathrm{D}})^\omega \times \kappa_{\mathrm{D}}$ where λ is a certain cardinal invariant of $\mathcal{I}_\kappa$. For this purpose we need to develop some theory on (Kadec) submeasures in §3.

4.2 Preliminaries

Let φ be a submeasure on a set A. For $I \subset A$ we let φ_I denote the restriction of φ to $\mathcal{P}(I)$. Without loss of generality we may always assume that A is a cardinal number. Unless stated otherwise κ, λ, and μ denote cardinal numbers. Remember that a regular space Z (or a topology) is called *zero-dimensional* if there is a basis consisting of clopen sets, see Definition 1.1.14.

Definition 4.2.1. Given $\psi : Z \to [0,\infty]$, define the following subspaces of $Z \times [0,\infty]$:

$$\begin{aligned} G_\psi^\infty &= \{(z,\psi(z)) : z \in Z,\ \psi(z) < \infty\}, \\ L_\psi^\infty &= \{(z,t) : z \in Z,\ \psi(z) \le t \le \infty\}. \end{aligned}$$

If Z is not empty, zero-dimensional, separable, and metrizable, then an LSC function ψ is said to be an *L-Lelek function* if G_ψ^∞ is dense in L_ψ^∞.

We have the following result, see Dijkstra, van Mill and Valkenburg [13, Lemma 22].

Lemma 4.2.2. *Let $\varepsilon > 0$ be given. If $\varphi\colon C \to [0,\infty]$ and $\psi\colon D \to [0,\infty]$ are L-Lelek functions with compact domain and if $\varphi^{-1}(0)$ and $\psi^{-1}(0)$ are singletons, then there are a homeomorphism $h\colon C \to D$ and a continuous $f\colon C \to (0,\infty)$ such that $\psi \circ h = f \cdot \varphi$ and* $\sup\{|\log f(x)| : x \in C\} < \varepsilon$.

Definition 4.2.3. The *weight* of a space Z is given by

$$w(Z) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a basis for the topology of } Z\} + \omega$$

and the *local weight* is given by

$$lw(Z) = \min\{w(U) : U \text{ is an open nonempty subset of } Z\}.$$

Definition 4.2.4. Let κ be an arbitrary infinite cardinal number, and let $p \ge 1$. We define the (possibly nonseparable) Banach space ℓ_κ^p, given by

$$\ell_\kappa^p = \left\{ x = (x_\alpha)_{\alpha\in\kappa} \in \mathbb{R}^\kappa : \sum_{\alpha\in\kappa} |x_\alpha|^p < \infty \right\},$$

equipped with the topology generated by the norm $\|x\|_p = (\sum_{\alpha\in\kappa} |x_\alpha|^p)^{1/p}$.

Remark 4.2.5. Recall that the norm on $\ell^p = \ell_\omega^p$ is a Kadec norm, that is, the norm topology is the weakest topology that makes all coordinate projections and the norm function continuous, see Proposition 1.2.1. The same holds for the norm on ℓ_κ^p. Thus, the graph of the norm function when seen as a function from ℓ_κ^p with the product topology (or any other topology that lies between the product topology and the norm topology) to $\mathbb{R}$ is homeomorphic to ℓ_κ^p by the obvious map.

Definition 4.2.6. For $\omega \le \lambda \le \kappa$, let

$$F_\alpha = \begin{cases} \{0\} \cup \{1/n : n \in \mathbb{N}\}, & \text{if } \alpha \in \lambda; \\ \{0,1\}, & \text{if } \alpha \in \kappa \setminus \lambda. \end{cases}$$

We define $\mathfrak{E}^1_{\mathrm{c}}(\lambda,\kappa) = \{x \in \ell^1_\kappa : x_\alpha \in F_\alpha, \alpha \in \kappa\}$.

By Dijkstra, van Mill and Valkenburg [13, Proposition 13] we have $lw(\mathfrak{E}^1_{\mathrm{c}}(\lambda,\kappa)) = \lambda$ and $w(\mathfrak{E}^1_{\mathrm{c}}(\lambda,\kappa)) = \kappa$. It is clear that $\mathfrak{E}^1_{\mathrm{c}}(\lambda,\kappa)$ is complete as a closed subset of ℓ^1_κ. For $\lambda = \kappa = \omega$, this space represents $\mathfrak{E}_{\mathrm{c}}$. Therefore, we will also refer to $\mathfrak{E}^1_{\mathrm{c}}(\lambda,\kappa)$ as a *nonseparable complete Erdős space* if $\kappa > \omega$. Remember that Erdős proved that both $\mathfrak{E}$ and $\mathfrak{E}_{\mathrm{c}}$ are one-dimensional spaces, yet they are totally disconnected and homeomorphic to their own squares. Nonseparable complete Erdős spaces have analogous properties, as shown by Dijkstra, van Mill and Valkenburg [13]. That paper also concerns more general spaces defined by

$$\mathcal{E}_\mu = \{x \in \ell^p_\mu : x_\alpha \in E_\alpha,\ \alpha \in \mu\},$$

where μ is another arbitrary infinite cardinal number and the sets E_α are arbitrary subsets of $\mathbb{R}$. The values of the cardinal invariants weight and local weight can easily be determined for the space $\mathcal{E}_\mu$ whenever the sets E_α are given as is shown by Dijkstra, van Mill and Valkenburg [13, Proposition 13]. The following theorem is the main result in [13].

Theorem 4.2.7. *The space $\mathcal{E}_\mu$ is homeomorphic to*

$$\mathfrak{E}_{\mathrm{c}} \times (\lambda_{\mathrm{D}})^\omega \times \kappa_{\mathrm{D}},$$

with $\lambda = lw(\mathcal{E}_\mu)$ and $\kappa = w(\mathcal{E}_\mu)$ if and only if $\operatorname{ind} \mathcal{E}_\mu > 0$ *and every E_α is a zero-dimensional G_δ-subset of* $\mathbb{R}$.

An easy consequence of this is [13, Theorem 41]:

Theorem 4.2.8. *Let $\omega \le \lambda \le \kappa$. Then $\mathfrak{E}^1_{\mathrm{c}}(\lambda,\kappa)$ is homeomorphic to $\mathfrak{E}_{\mathrm{c}} \times (\lambda_{\mathrm{D}})^\omega \times \kappa_{\mathrm{D}}$.*

4.3 Basic properties and Kadec submeasures

We start with a couple of useful relations between the theory of submeasures on ω, as given in Chapter 3, and submeasures on an arbitrary cardinal κ. Subsequently we introduce the notion of a Kadec submeasure on an arbitrary cardinal κ and we discuss some applications. The following lemma gives another proof of the fact that elements of the exhaustive ideal of a submeasure are at most countable, see Lemma 1.1.6.

Lemma 4.3.1. *Let φ be a submeasure on $\kappa > \omega$. Then*

$$\begin{aligned} \mathrm{Exh}(\varphi) &= \{X \subset \kappa : X \cap I \in \mathrm{Exh}(\varphi_I) \text{ for all } I \subset \kappa \text{ with } |I| = \omega\} \\ &= \bigcup_{I \subset \kappa,\, |I|=\omega} \mathrm{Exh}(\varphi_I) \end{aligned}$$

and hence every element of $\mathrm{Exh}(\varphi)$ *is at most countable.*

Proof. Let $\mathcal{A}$ denote the family of all subsets X of κ with the property that $X \cap I \in \mathrm{Exh}(\varphi_I)$ for all $I \subset \kappa$ with $|I| = \omega$. Let $X \in \mathrm{Exh}(\varphi)$ and let $\varepsilon > 0$. We can find a finite subset F of κ such that $\varphi(X \setminus F) < \varepsilon$. Now take $I \subset \kappa$ such that $|I| = \omega$. Then $I \cap F$ is a finite subset of I and $\varphi((X \cap I) \setminus (F \cap I)) = \varphi((X \setminus F) \cap I) \leq \varphi(X \setminus F) < \varepsilon$. This means that $X \cap I \in \mathrm{Exh}(\varphi_I)$, so $\mathrm{Exh}(\varphi) \subset \mathcal{A}$.

The remaining inclusions become trivial once we show that every $X \in \mathcal{A}$ is at most countable. Take $X \in \mathcal{A}$ and let $n \in \mathbb{N}$. Suppose that $X_n = \{\alpha \in X : \varphi(\{\alpha\}) > 1/n\}$ is an infinite set. Then, by definition of $\mathcal{A}$, we can find a countable infinite set $I \subset X_n$ such that $I \in \mathrm{Exh}(\varphi_I)$. This is a contradiction since, by the monotonicity of φ, we have $\varphi(I \setminus F) > 1/n$ for every finite subset F of I. Hence $|X_n| < \omega$ and we conclude that $X = \bigcup_{m=1}^{\infty} X_m$ is indeed at most countable. □

The following theorem generalizes Theorem 3.2.3 and again follows easily from Solecki [29].

Theorem 4.3.2. *Suppose that φ is an LSC submeasure on κ, then*

$$d_\varphi(X, Y) = \varphi(X \triangle Y)$$

defines an invariant, complete metric on $\mathrm{Exh}(\varphi)$ *and an invariant metric on* $\mathrm{Fin}(\varphi)$.

Proof. We will verify that d_φ is complete on $\mathrm{Exh}(\varphi)$. Consider a Cauchy sequence $(X_n)_{n\in\omega}$ in $\mathrm{Exh}(\varphi)$ and let $I = \bigcup_{n\in\omega} X_n$. Then according to Solecki [29, Theorem 3.1] d_{φ_I} is a complete metric on $\mathrm{Exh}(\varphi_I)$, hence there exists an $X \in \mathrm{Exh}(\varphi_I)$ with $d_{\varphi_I}(X_n, X) \to 0$ as $n \to \infty$. Now observe that $d_{\varphi_I}(X_n, X) = d_\varphi(X_n, X)$, which implies that $d_\varphi(X_n, X) \to 0$ as $n \to \infty$. Since $X \in \mathrm{Exh}(\varphi)$ by Lemma 4.3.1, the proof is complete. $\square$

We will write d instead of d_φ whenever there is no danger of confusion. Observe that the topology τ_d generated by the metric d on $\mathrm{Exh}(\varphi)$ and $\mathrm{Fin}(\varphi)$ is stronger than the product topology τ_{w} that these spaces inherit from 2^κ. Also note that $(\mathrm{Exh}(\varphi), \tau_d)$ and $(\mathrm{Fin}(\varphi), \tau_d)$ are topological groups with the symmetric difference '$\triangle$' as group operation.

Notation 4.3.3. As in Theorem 4.1.1 and Theorem 4.1.2 we write $\mathcal{I}_A = (\mathrm{Exh}(\varphi), \tau_d)$ for an LSC submeasure φ on a set A, where τ_d is the topology generated by the metric d of Theorem 4.3.2. For $B \subset A$ we write $\mathcal{I}_B$ for $(\mathrm{Exh}(\varphi_B), \tau_{d_{\varphi_B}})$.

Apart from the topology τ_d on $\mathrm{Exh}(\varphi)$ for a submeasure φ, one can also consider another natural topology.

Definition 4.3.4. Let (Z, τ) and (W, τ') be topological spaces and $f\colon Z \to W$ a map. Then τ_f is the weakest topology on Z that contains τ and that makes f continuous.

We shall simply denote the topology that a subset of Z inherits from $\big(Z, \tau_f\big)$ by τ_f and the topology it inherits from $\big(Z, \tau\big)$ by τ. If $\mathcal{S}$ is a basis for the topology τ' on W that is closed under finite intersections, then the collection $\mathcal{B}$ given by

$$\mathcal{B} = \{O \cap f^{-1}(S) : O \in \tau \text{ and } S \in \mathcal{S}\}$$

forms a basis for τ_f. If $W = [0, \infty]$ and f is LSC and we restrict ourselves to the set $f^{-1}([0, \infty))$, then the collection $\mathcal{C}$ given by

$$\mathcal{C} = \{O \cap f^{-1}\big([0, t)\big) : O \in \tau \text{ and } t \in (0, \infty)\}$$

forms a basis for τ_f on $f^{-1}\big([0, \infty)\big)$.

Recall that the graph of $f\colon Z \to W$ is the set of pairs $\{(z, f(z)) : z \in Z\}$ in the product space $Z \times W$.

Remark 4.3.5. Just as for the norm on ℓ^p_κ (see Remark 4.2.5), the graph of f when seen as a function from (Z, τ) (or Z with any other topology that lies between τ and τ_f) to W is homeomorphic to (Z, τ_f) by the obvious map.

Note that if we take f to be an LSC submeasure on κ, then $Z = 2^\kappa$ with the product topology τ_{w} and $W = [0, \infty]$. The following lemma is a generalization of Lemma 3.3.11.

Lemma 4.3.6. *Let φ be an LSC submeasure on κ. Then we have the following relation between the topologies $\tau_{\mathrm{w}}, \tau_\varphi$ and τ_d on $\mathrm{Exh}(\varphi)$: $\tau_{\mathrm{w}} \subset \tau_\varphi \subset \tau_d$.*

Proof. Exactly the same as the proof of Lemma 3.3.11, just replace ω by κ. □

It is a natural question to ask when $\tau_\varphi = \tau_d$ on $\mathrm{Exh}(\varphi)$. The following definition extends Definition 3.3.12.

Definition 4.3.7. An LSC submeasure φ on κ is called a *Kadec submeasure* if τ_φ equals the group topology τ_d on $\mathrm{Exh}(\varphi)$.

We have the following result, which generalizes Proposition 3.3.13.

Proposition 4.3.8. *An LSC measure φ on κ is a Kadec submeasure.*

Proof. Exactly the same as the proof of Proposition 3.3.13, just replace ω by κ and replace references to Lemma 3.3.11 by Lemma 4.3.6 and Theorem 3.2.3 by Theorem 4.3.2. □

Let for $A \subset \kappa$ the function $\zeta_A \colon 2^\kappa \to 2^\kappa$ be given by $\zeta_A(X) = X \setminus A$. The following result generalizes Proposition 3.3.14.

Proposition 4.3.9. *Let φ be an LSC submeasure on κ. The following statements are equivalent:*

(1) *φ is a Kadec submeasure;*

(2) *$\varphi \circ \zeta_{\{\alpha\}} \colon (\mathrm{Exh}(\varphi), \tau_\varphi) \to \mathbb{R}$ is continuous for every $\alpha \in \kappa$; and*

(3) *$\zeta_{\{\alpha\}} \colon (\mathrm{Exh}(\varphi), \tau_\varphi) \to (\mathrm{Exh}(\varphi), \tau_\varphi)$ is continuous for each $\alpha \in \kappa$.*

Proof. The proof is the same as the proof of Proposition 3.3.14, just replace ω by κ and replace references to Lemma 3.3.11 by Lemma 4.3.6 and Theorem 3.2.3 by Theorem 4.3.2. $\square$

The following general result will be useful in different places in the remainder of this chapter.

Lemma 4.3.10. *Let (Z, τ) be a topological space and let $\varphi_i \colon Z \to [0, \infty]$ be LSC functions for $i \in I$, where I is an arbitrary nonempty set. Define $\varphi \colon Z \to [0, \infty]$ by $\varphi(z) = \sum_{i \in I} \varphi_i(z)$. Then φ is an LSC function and on $\varphi^{-1}\bigl([0, \infty)\bigr)$ the topology τ_{φ_i} is weaker than τ_φ for every $i \in I$.*

Proof. It is easily verified that φ is LSC, using the fact that every φ_i is nonnegative. Now we show that $\tau_{\varphi_i} \subset \tau_\varphi$ on $\varphi^{-1}\bigl([0, \infty)\bigr)$ for every $i \in I$. Pick $i_0 \in I$. As observed before, the family

$$\mathcal{B}_\varphi = \{O \cap \varphi^{-1}\bigl([0, t)\bigr) : O \in \tau \text{ and } t \in (0, \infty)\}$$

is a basis for τ_φ on $\varphi^{-1}\bigl([0, \infty)\bigr)$. Similarly, since $\varphi^{-1}\bigl([0, \infty)\bigr) \subset \varphi_{i_0}^{-1}\bigl([0, \infty)\bigr)$, the family

$$\mathcal{B}_{\varphi_{i_0}} = \{O \cap \varphi_{i_0}^{-1}\bigl([0, t)\bigr) : O \in \tau \text{ on } \varphi^{-1}\bigl([0, \infty)\bigr) \text{ and } t \in (0, \infty)\}$$

is a basis for $\tau_{\varphi_{i_0}}$ on $\varphi^{-1}\bigl([0, \infty)\bigr)$. Hence it is sufficient to show that

$$\varphi_{i_0}^{-1}\bigl([0, t)\bigr) \cap \varphi^{-1}\bigl([0, \infty)\bigr) \in \tau_\varphi$$

for every $t \in (0, \infty)$.

Pick $t \in (0, \infty)$. If $\varphi_{i_0}^{-1}\bigl([0, t)\bigr) \cap \varphi^{-1}\bigl([0, \infty)\bigr) = \emptyset$, there is nothing to prove, so suppose $z \in Z$ is such that $\varphi_{i_0}(z) < t$ and $\varphi(z) < \infty$. Note that

$$\varphi = \varphi_{i_0} + \psi,$$

with $\psi = \sum_{i \in I \setminus \{i_0\}} \varphi_i$. Observe that ψ is also an LSC function on Z and we have $\psi(z) \le \varphi(z) < \infty$. Put $\delta = t - \varphi_{i_0}(z) > 0$ and consider the set $V \subset Z$ given by

$$V = \psi^{-1}\bigl((\psi(z) - \delta/2, \infty]\bigr) \cap \varphi^{-1}\bigl([0, \varphi(z) + \delta/2)\bigr).$$

Clearly, $z \in V \in \mathcal{B}_\varphi$. Take $y \in V$, then $\varphi_{i_0}(y)+\psi(y) < \varphi_{i_0}(z)+\psi(z)+\delta/2$ and $\psi(y) > \psi(z) - \delta/2$. It follows that

$$\varphi_{i_0}(y) < \varphi_{i_0}(z) + \psi(z) - \psi(y) + \delta/2 < \varphi_{i_0}(z) + \delta = t.$$

This means that $z \in V \subset \varphi_{i_0}^{-1}\big([0,t)\big) \cap \varphi^{-1}\big([0,\infty)\big)$ which finishes the proof. $\square$

Corollary 4.3.11. *Let $\{A_i : i \in I\}$ be a nonempty collection of pairwise disjoint sets and write $A = \bigcup_{i\in I} A_i$. Suppose that $\varphi_i \colon 2^{A_i} \to [0,\infty]$ is a Kadec submeasure for each $i \in I$. Then the function $\varphi\colon 2^A \to [0,\infty]$ defined by*

$$\varphi(X) = \sum_{i\in I} \varphi_i(X \cap A_i)$$

is a Kadec submeasure on A.

Proof. If $|I| = 1$ there is nothing to prove, so suppose that $|I| > 1$. We will use Lemma 4.3.10 and Proposition 4.3.9 to show that φ is a Kadec submeasure.

We extend φ_i to an LSC function $\tilde\varphi_i$ on 2^A in the obvious way: we define $\tilde\varphi_i \colon 2^A \to [0,\infty]$ by

$$\tilde\varphi_i(X) = \varphi_i(X \cap A_i).$$

Since $\varphi = \sum_{i\in I} \tilde\varphi_i$ it follows from Lemma 4.3.10 that φ is an LSC function on 2^A. Furthermore, since every φ_i is a submeasure on A_i, it is easily seen that φ is a submeasure on A.

Take $\alpha \in A$, say $\alpha \in A_j$ for some $j \in I$. We show that $\varphi \circ \zeta_{\{\alpha\}} \colon (\mathrm{Exh}(\varphi), \tau_\varphi) \to \mathbb{R}$ is continuous, which means that φ is Kadec by statement (2) of Proposition 4.3.9. Define $\psi\colon 2^A \to [0,\infty]$ by

$$\psi(X) = \sum_{i \in I\setminus\{j\}} \tilde\varphi_i(X).$$

By Lemma 4.3.10 we know that ψ is an LSC function and we have $\varphi = \tilde\varphi_j + \psi$ and $\varphi\circ\zeta_{\{\alpha\}} = \tilde\varphi_j \circ \zeta_{\{\alpha\}} + \psi$. Furthermore, Lemma 4.3.10 tells us that $\tau_{\tilde\varphi_j} \subset \tau_\varphi$ and $\tau_\psi \subset \tau_\varphi$ on $\mathrm{Fin}(\varphi) \supset \mathrm{Exh}(\varphi)$. Since ψ is clearly continuous on $(\mathrm{Exh}(\varphi), \tau_\psi)$ this implies that ψ is continuous on

$(\mathrm{Exh}(\varphi), \tau_\varphi)$. We are done once we show that $\tilde{\varphi}_j \circ \zeta_{\{\alpha\}} : (\mathrm{Exh}(\varphi), \tau_{\tilde{\varphi}_j}) \to \mathbb{R}$ is continuous. This follows from the continuity of the projection $X \mapsto X \cap A_j$ as a function from $(\mathrm{Exh}(\varphi), \tau_{\tilde{\varphi}_j})$ to $(\mathrm{Exh}(\varphi), \tau_{\varphi_j})$ together with the continuity of $\varphi_j \circ \zeta_{\{\alpha\}}$ on $(\mathrm{Exh}(\varphi_j), \tau_{\varphi_j})$, see Proposition 4.3.9. □

As mentioned before Lemma 3.3.11 there are LSC submeasures that are not Kadec submeasures. However, for LSC submeasures on $\kappa \leq \omega$ we have the following theorem, which is a combination of Theorem 3.3.15 and the fact that submeasures on finite sets are automatically Kadec submeasures.

Theorem 4.3.12. *Let φ be an LSC submeasure on $\kappa \leq \omega$. Then there exists a Kadec submeasure ψ such that $\varphi \leq \psi \leq 2\varphi$.*

As we noted in Remark 3.3.16 we clearly have $\mathrm{Exh}(\psi) = \mathrm{Exh}(\varphi)$ and $\mathrm{Fin}(\psi) = \mathrm{Fin}(\varphi)$ and it is also clear that the metrics d_φ and d_ψ as in Theorem 4.3.2 are uniformly equivalent. This means for example that we may assume without loss of generality that the submeasure φ in Theorem 4.1.1 is a Kadec submeasure, an observation that we used extensively in Chapter 3. Unfortunately, the proof of Theorem 3.3.15 does not generalize to LSC submeasures on an uncountable cardinal number, so we can ask the following question.

Question 4.3.13. Let φ be an LSC submeasure on $\kappa > \omega$. Is there a Kadec submeasure ψ on κ with $\mathrm{Exh}(\psi) = \mathrm{Exh}(\varphi)$ and such that $\tau_{d_\varphi} = \tau_\psi$ on $\mathrm{Exh}(\varphi)$?

4.4 The small inductive dimension of $\mathfrak{I}_\kappa$

We now return to the setting of general LSC submeasures. In this section we will prove Theorem 4.1.2, the extension of Theorem 4.1.1 to uncountable cardinals, and consider some consequences. Among them are the relation between the minimal weight of nonempty open subsets of $\mathfrak{I}_\kappa$ and the small inductive dimension, and a statement concerning a one-point connectification of $\mathfrak{I}_\kappa$.

We start by proving Theorem 4.1.2.

Proof of **Theorem 4.1.2.** (1) $\Rightarrow$ (2). We prove this implication by contraposition. Suppose there exists a subset B of $\kappa > \omega$ with

$\mathrm{Exh}(\varphi) = \{X \subset \kappa : X \cap B \text{ is finite}\}$. Since $\kappa \setminus B$ is in the collection, we either have $|\kappa \setminus B| < \omega$ or $|\kappa \setminus B| = \omega$. If $|\kappa \setminus B| < \omega$, then $\mathrm{Exh}(\varphi)$ consists of all finite subsets of κ, thus $|\mathrm{Exh}(\varphi)| = \kappa$. Note that $\inf_{\alpha\in\kappa} \varphi(\{\alpha\}) \neq 0$, for otherwise there exists a sequence $\{\alpha_i\}_{i\in\omega}$ with $\lim_{i\in\omega} \varphi(\{\alpha_i\}) = 0$. Then we can find a subsequence $\{\alpha_{i_k}\}_{k\in\omega}$ for which $\varphi(\{\alpha_{i_k}\}) \leq 2^{-k}$ and hence $\varphi(\{\alpha_{i_1}, \alpha_{i_2}, \ldots\}) \leq 2$ by Lemma 1.1.10. This means that there would be an infinite set in $\mathrm{Exh}(\varphi)$, which leads to a contradiction. Since for all finite sets $F \neq F'$ we have $d(F, F') \geq \inf_{\alpha\in\kappa} \varphi(\{\alpha\})$, every finite set is isolated in $\mathcal{I}_\kappa$. We see that the space $\mathcal{I}_\kappa$ is homeomorphic to κ_{D}. If $|\kappa \setminus B| = \omega$, we will show that

$$\mathcal{I}_\kappa \approx \bigoplus_{F\subset B,\ F \text{ finite}} (\{F \cup X : X \subset \kappa \setminus B\}, \tau_d) \approx \kappa_{\mathrm{D}} \times 2^\omega.$$

Let F and F' be finite subsets of B and let $X, X' \subset \kappa \setminus B$. Then $d(F \cup X, F' \cup X') = \varphi((F \triangle F') \cup (X \triangle X')) \geq \varphi(F \triangle F')$. As above, it follows for $F \neq F'$ that $d(F, F') \geq \inf_{\alpha\in B} \varphi(\{\alpha\}) > 0$. Thus all collections $\{F \cup X : X \subset \kappa \setminus B\}$ are clopen in $\mathcal{I}_\kappa$. Clearly, $(\{F \cup X : X \subset \kappa \setminus B\}, \tau_d) \approx (\mathcal{P}(\kappa \setminus B), \tau_d)$, where $\kappa \setminus B$ is countably infinite. Note that $\mathrm{Exh}(\varphi_{\kappa\setminus B}) = \mathrm{Fin}(\varphi_{\kappa\setminus B}) = \mathcal{P}(\kappa \setminus B)$, so it follows from Theorem 3.2.3 that τ_d is a Polish group topology on $\mathcal{P}(\kappa \setminus B)$ that is stronger than τ_{w}. Since τ_{w} is obviously a Polish group topology on $\mathcal{P}(\kappa \setminus B)$ that witnesses the fact that $\mathcal{P}(\kappa \setminus B)$ is a Polishable ideal, we can apply Proposition 3.2.2 to obtain that $\tau_d = \tau_{\mathrm{w}}$ on $\mathcal{P}(\kappa \setminus B)$. We see that $(\mathcal{P}(\kappa \setminus B), \tau_d) \approx 2^\omega$ and we conclude that $\mathcal{I}_\kappa \approx \kappa_{\mathrm{D}} \times 2^\omega$.

(2) $\Rightarrow$ (3). We prove this implication by contraposition again. Suppose there exists an $\varepsilon > 0$ for which $\{\alpha \in \kappa : \varphi(\{\alpha\}) \leq \varepsilon\} \in \mathrm{Exh}(\varphi)$. Then for $B = \{\alpha \in \kappa : \varphi(\{\alpha\}) > \varepsilon\}$ we have $\mathrm{Exh}(\varphi) = \{X \subset \kappa : X \cap B \text{ is finite}\}$. Indeed, if $X \cap B$ is finite, then it is contained in $\mathrm{Exh}(\varphi)$, as is $\kappa \setminus B$. Since $X \subset (X \cap B) \cup (\kappa \setminus B)$ we have $X \in \mathrm{Exh}(\varphi)$ as well. If $X \in \mathrm{Exh}(\varphi)$, then there is a finite set F with $\varphi(X \setminus F) \leq \varepsilon$. This gives $X \setminus F \subset \kappa \setminus B$ and therefore $X \cap B = X \setminus (\kappa \setminus B) \subset F$ is finite.

(3) $\Rightarrow$ (4). Assume (3). Take $m \in \mathbb{N}$ and let $X_m = \{\alpha \in \kappa : \varphi(\{\alpha\}) \leq 1/m\}$. Then we know that $X_m \notin \mathrm{Fin}(\varphi)$, so since φ is LSC it follows that there is a countable subset $I(m)$ of X_m such that $\varphi(I(m)) = \infty$. Take $I = \bigcup_{m=1}^\infty I(m)$. Then $|I| = \omega$ and for every $m \in \mathbb{N}$ we see

that $\varphi(\{\alpha \in I : \varphi(\{\alpha\}) \le 1/m\}) \ge \varphi(I(m)) = \infty$. Now look at φ_I and note that $\mathrm{Exh}(\varphi_I) = \mathrm{Fin}(\varphi_I)$. Applying Theorem 4.1.1 we see that there exists a set $B \subset I$ such that $\varphi_I(B) = \infty$ and $\lim_{\alpha \in B} \varphi_I(\{\alpha\}) = 0$. We conclude that B satisfies condition (4).

(4) ⇒ (5). Suppose that B satisfies (4) and consider φ_B. Note that $|B| = \omega$. Since $\mathrm{Exh}(\varphi_B) = \mathrm{Fin}(\varphi_B)$ we can use Theorem 4.1.1 to find that $\mathfrak{I}_B \approx \mathfrak{E}_c$. Furthermore, since

$$\begin{aligned} d_\varphi(X,Y) = \varphi(X \triangle Y) &\le \varphi_B((X \triangle Y) \cap B) + \varphi_{\kappa \setminus B}((X \triangle Y) \setminus B) \\ &= d_{\varphi_B}(X \cap B, Y \cap B) + d_{\varphi_{\kappa \setminus B}}(X \setminus B, Y \setminus B) \le 2 d_\varphi(X,Y), \end{aligned}$$

we find (with the corresponding metric topologies) that

$$\mathfrak{I}_\kappa \times \mathfrak{E}_c \approx \mathfrak{I}_{\kappa\setminus B} \times \mathfrak{I}_B \times \mathfrak{E}_c \approx \mathfrak{I}_{\kappa\setminus B} \times \mathfrak{E}_c^2 \approx \mathfrak{I}_{\kappa\setminus B} \times \mathfrak{E}_c \approx \mathfrak{I}_{\kappa\setminus B} \times \mathfrak{I}_B \approx \mathfrak{I}_\kappa.$$

(4) ⇒ (6). Suppose that $B \subset \kappa$ satisfies condition (4) and hence $|B| = \omega$. Take a nonempty clopen subset C of $\mathfrak{I}_\kappa$ and let the set X be an element of C. Put $Y = B \cup X$ and note that $|Y| = \omega$ and $\mathrm{Exh}(\varphi_Y) = \mathrm{Fin}(\varphi_Y)$. Then $X \in C \cap \mathfrak{I}_Y$, so we have that $C \cap \mathfrak{I}_Y$ is a nonempty clopen subset of $\mathfrak{I}_Y$. Since $B \subset Y$ it follows from Theorem 4.1.1 that $C \cap \mathfrak{I}_Y$ is φ_Y-unbounded and hence C is φ-unbounded.

The implications (5) ⇒ (7) and (6) ⇒ (7) are trivial. For (7) ⇒ (8) note that $\mathfrak{I}_\kappa$ is totally disconnected and that a totally disconnected locally compact space is zero-dimensional. The implication (8) ⇒ (1) is trivial. □

Remark 4.4.1. Note that in proving (4) ⇒ (5) we first showed that it follows from (4) that

(9) there is a $B \subset \kappa$ with $|B| = \omega$ and $\mathfrak{I}_B$ is homeomorphic to $\mathfrak{E}_c$.

Then we showed that this implies statement (5). So we can add statement (9) to the list of equivalences in Theorem 4.1.2.

Remember that if Z is a nonempty space then W is called a *Z-factor* if there is a space T such that $W \times T$ is homeomorphic to Z, see also the remark before Theorem 3.5.1. So statement (5) implies that $\mathfrak{E}_c$ is an $\mathfrak{I}_\kappa$-factor. It is an easy exercise, using the equivalence of (5) and (7) in Theorem 4.1.2, to show that we have the following additional equivalences in that theorem:

(10) $\mathfrak{E}_c$ is an $\mathfrak{I}_\kappa$-factor;

(11) $\mathfrak{E}_c$ is homeomorphic to a retract of $\mathfrak{I}_\kappa$;

(12) there exists a closed embedding of $\mathfrak{E}_c$ in $\mathfrak{I}_\kappa$; and

(13) there exists an embedding of $\mathfrak{E}_c$ in $\mathfrak{I}_\kappa$.

Remark 4.4.2. For a Kadec submeasure φ on $\kappa > \omega$ with $\mathrm{Exh}(\varphi) = \mathrm{Fin}(\varphi)$ one can view the space $\mathfrak{I}_\kappa$ as the graph of φ when seen as a function from $(\mathrm{Exh}(\varphi), \tau_{\mathrm{w}})$ to $[0, \infty)$. Since $\mathrm{ind}((\mathrm{Exh}(\varphi), \tau_{\mathrm{w}}) \times [0, \infty)) = 1$ we can replace statement (7) in Theorem 4.1.2 in this case by the statement that $\mathrm{ind}\mathfrak{I}_\kappa = 1$. In particular, if the answer to Question 4.3.13 is in the affirmative, we can reformulate statement (7) in Theorem 4.1.2 in general.

Remark 4.4.3. The condition '$\mathrm{Exh}(\varphi) = \mathrm{Fin}(\varphi)$' is really needed in Theorem 4.1.2 as the following examples show. In [10, Example 42] Dijkstra and van Mill consider an LSC submeasure φ_1 on $\omega \times \omega$ with $\mathrm{Exh}(\varphi_1) \neq \mathrm{Fin}(\varphi_1)$ such that $\mathfrak{I}_{\omega\times\omega}$ is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$. Take a cardinal number $\kappa > \omega$ and define the LSC submeasure ψ on $(\omega \times \omega) \cup \kappa$ by

$$\psi(X \cup Y) = \varphi_1(X) + \sum_{\alpha \in Y} 1,$$

where $X \subset \omega \times \omega$ and $Y \subset \kappa$. Since $\mathfrak{I} = \mathfrak{I}_{(\omega\times\omega)\cup\kappa}$ is homeomorphic to $\mathfrak{I}_{\omega\times\omega} \times \mathfrak{I}_\kappa$ we see that $\mathfrak{I}$ is homeomorphic to $\mathbb{R} \setminus \mathbb{Q} \times \kappa_{\mathrm{D}}$. This means that $\mathfrak{I}$ is zero-dimensional and not homeomorphic to κ_{D} or $\kappa_{\mathrm{D}} \times 2^\omega$. We see that statement (1) of Theorem 4.1.2 does not imply statement (7) in this case.

In the same example Dijkstra and van Mill introduce an LSC submeasure φ_3 on $\omega \times \omega$ with $\mathrm{Exh}(\varphi_3) \neq \mathrm{Fin}(\varphi_3)$ such that $\mathfrak{I}_{\omega\times\omega}$ is homeomorphic to $\mathfrak{E}_c^\omega$ and $\varphi_3 \leq 2$. For a cardinal number $\kappa > \omega$ we now define the LSC submeasure ψ on $(\omega \times \omega) \cup \kappa$ by

$$\psi(X \cup Y) = \varphi_3(X) + \sum_{\alpha \in Y} 1,$$

where $X \subset \omega \times \omega$ and $Y \subset \kappa$. In the same way as in the previous case we see that $\mathfrak{I} = \mathfrak{I}_{(\omega\times\omega)\cup\kappa}$ is homeomorphic to $\mathfrak{E}_c^\omega \times \kappa_{\mathrm{D}}$ and hence

$\operatorname{ind}\mathfrak{I} > 0$. However, since $\varphi_3 \le 2$ we cannot find a $B \subset (\omega \times \omega) \cup \kappa$ such that $\psi(B) = \infty$ and $\lim_{\alpha \in B} \varphi(\{\alpha\}) = 0$. We see that statement (7) of Theorem 4.1.2 does not imply statement (4) in this case.

Proposition 4.4.4. *Let φ be an LSC submeasure on $\kappa \ge \omega$. Then $w\big(\mathfrak{I}_\kappa\big) = \kappa$ and $lw\big(\mathfrak{I}_\kappa\big) = \min_{n\in\mathbb{N}} |\{\alpha \in \kappa : \varphi(\{\alpha\}) < 1/n\}| + \omega$.*

PROOF. The collection of all finite subsets of κ is dense in $\mathfrak{I}_\kappa$, by definition of $\operatorname{Exh}(\varphi)$. Therefore $w\big(\mathfrak{I}_\kappa\big) \le \kappa$. The collection of all singletons forms a discrete subset of $\mathfrak{I}_\kappa$ of cardinality κ: for arbitrary $\alpha \in \kappa$ and $\beta \ne \alpha$ we have $d(\{\alpha\},\{\beta\}) = \varphi(\{\alpha,\beta\}) \ge \varphi(\{\alpha\}) > 0$ and hence $w\big(\mathfrak{I}_\kappa\big) \ge \kappa$.

Since we have a topological group structure, it suffices to show for all $n \in \mathbb{N}$ that $w\big(\varphi^{-1}([0,1/n)) \cap \mathfrak{I}_\kappa\big) = |L| + \omega$, where $L = \{\alpha \in \kappa : \varphi(\{\alpha\}) < 1/n\}$. The same reasoning as above yields that $w\big(\varphi^{-1}([0,1/n)) \cap \mathfrak{I}_\kappa\big) \ge |L| + \omega$, so it is left to prove that we can reverse the inequality. We know that the collection $\mathfrak{F}$ of finite sets in $\varphi^{-1}([0,1/n)) \cap \mathfrak{I}_\kappa$ forms a dense set in $\varphi^{-1}([0,1/n)) \cap \mathfrak{I}_\kappa$. Let $\mathfrak{F}_L$ denote the collection of finite subsets of L. We have $\mathfrak{F} \subset \mathfrak{F}_L$, which means that $w\big(\varphi^{-1}([0,1/n)) \cap \mathfrak{I}_\kappa\big) \le |\mathfrak{F}| + \omega \le |\mathfrak{F}_L| + \omega = |L| + \omega$. □

Example 4.4.5. Define the following LSC measure on $\omega \times \kappa$ for some infinite cardinal number κ and another infinite cardinal number $\lambda \le \kappa$:

$$\eta(X) = \sum_{(n,\alpha)\in X : \alpha \in \lambda} 2^{-n} + \sum_{(n,\alpha)\in X : \alpha \notin \lambda} 1.$$

Since η is an LSC measure, we have $\operatorname{Exh}(\eta) = \operatorname{Fin}(\eta)$. Note that all statements in Theorem 4.1.2 hold. Furthermore, by Proposition 4.4.4 we find that $w(\mathfrak{I}_{\omega\times\kappa}) = |\omega \times \kappa| = \kappa$ and $lw(\mathfrak{I}_{\omega\times\kappa}) = |\omega \times \lambda| = \lambda$, hence the weight and local weight do not necessarily coincide. We can consider the following embedding of $\mathfrak{I}_{\omega\times\kappa}$ in the Banach space $\ell^1_{\omega\times\kappa} \approx \ell^1_\kappa$:

$$\big(h(X)\big)_{(n,\alpha)} = \begin{cases} 2^{-n}, & \text{if } (n,\alpha) \in X \text{ and } \alpha \in \lambda; \\ 1, & \text{if } (n,\alpha) \in X \text{ and } \alpha \in \kappa \setminus \lambda; \\ 0, & \text{if } (n,\alpha) \notin X, \end{cases}$$

for $X \in \mathfrak{I}_{\omega\times\kappa}$. Then $\eta(X \triangle Y) = \|h(X) - h(Y)\|_1$. Using Theorem 4.2.7 it can be shown that $h(\mathfrak{I}_{\omega\times\kappa}) \approx \mathfrak{E}_{\mathrm{c}} \times (\lambda_{\mathrm{D}})^\omega \times \kappa_{\mathrm{D}}$.

Proposition 4.4.6. *Suppose that φ is an LSC submeasure on $\kappa > \omega$ with* $\mathrm{Exh}(\varphi) = \mathrm{Fin}(\varphi)$. *If $lw(\mathfrak{I}_\kappa) > \omega$ then* $\operatorname{ind}\mathfrak{I}_\kappa > 0$.

Proof. Proposition 4.4.4 yields that $\min_{n\in\mathbb{N}} |\{\alpha \in \kappa : \varphi(\{\alpha\}) < 1/n\}| > \omega$. Now the set $\{\alpha \in \kappa : \varphi(\{\alpha\}) \le \varepsilon\}$ is uncountable for every $\varepsilon > 0$ and hence it cannot be a member of $\mathrm{Exh}(\varphi)$. By Theorem 4.1.2 we find $\operatorname{ind}\mathfrak{I}_\kappa > 0$. □

Corollary 4.4.7. *Let φ be an LSC submeasure on $\kappa > \omega$ such that* $\mathrm{Exh}(\varphi) = \mathrm{Fin}(\varphi)$ *and* $\operatorname{ind}\mathfrak{I}_\kappa > 0$. *If $lw(\mathfrak{I}_\kappa) = \omega$ then $\mathfrak{I}_\kappa$ is homeomorphic to $\mathfrak{E}_\mathrm{c} \times \kappa_\mathrm{D}$.*

Proof. Since $\operatorname{ind}\mathfrak{I}_\kappa > 0$ it follows from statement (9) in Remark 4.4.1 that there is a $B \subset \kappa$ with $|B| = \omega$ and $\mathfrak{I}_B \approx \mathfrak{E}_\mathrm{c}$. Using Proposition 4.4.4 we take $n \in \mathbb{N}$ such that $|\{\alpha \in \kappa : \varphi(\{\alpha\}) < 1/n\}| + \omega = \omega$. Now we define the set $I \subset \kappa$ as $I = B \cup \{\alpha \in \kappa : \varphi(\{\alpha\}) < 1/n\}$. Note that $|I| = \omega$ and $\mathfrak{I}_I \supset \mathfrak{I}_B$ so $\operatorname{ind}\mathfrak{I}_I > 0$. It follows from Theorem 4.1.1 that $\mathfrak{I}_I \approx \mathfrak{E}_\mathrm{c}$. Clearly, $\mathfrak{I}_\kappa \approx \mathfrak{I}_I \times \mathfrak{I}_{\kappa\setminus I}$, so we have $\mathfrak{I}_\kappa \approx \mathfrak{E}_\mathrm{c} \times \mathfrak{I}_{\kappa\setminus I}$. If $\alpha \in \kappa \setminus I$ then $\varphi(\{\alpha\}) \ge 1/n$ and this means that $\mathfrak{I}_{\kappa\setminus I}$ is the collection of all finite subsets of $\kappa \setminus I$ with the discrete topology. Since $|\mathfrak{I}_{\kappa\setminus I}| = \kappa$, the space $\mathfrak{I}_{\kappa\setminus I}$ is homeomorphic to κ_D. We conclude that $\mathfrak{I}_\kappa \approx \mathfrak{E}_\mathrm{c} \times \kappa_\mathrm{D}$. □

Remember the definition of a fixed point, see Definition 3.7.1. Let $\mathfrak{I}_\kappa^+ = \mathfrak{I}_\kappa \cup \{\Omega\}$ be a Hausdorff-extension of $\mathfrak{I}_\kappa$ such that for every neighbourhood U of Ω in $\mathfrak{I}_\kappa^+$ the complement $\mathfrak{I}_\kappa \setminus U$ is φ-bounded.

Theorem 4.4.8. *Suppose that φ is an LSC submeasure on $\kappa \ge \omega$ such that* $\mathrm{Exh}(\varphi) = \mathrm{Fin}(\varphi)$. *Then the following statements are equivalent:*

(1) *Ω is a fixed point of $\mathfrak{I}_\kappa^+$;*

(2) *$\mathfrak{I}_\kappa^+$ has the fixed point property;*

(3) *$\mathfrak{I}_\kappa^+$ is connected; and*

(4) $\operatorname{ind}\mathfrak{I}_\kappa > 0$.

Proof. The implications (1) ⇒ (2) and (2) ⇒ (3) are trivial.

(3) ⇒ (4). Assume that $\operatorname{ind}\mathfrak{I}_\kappa = 0$ and select a set $X \in \mathfrak{I}_\kappa$. Let U and V be disjoint and open in $\mathfrak{I}_\kappa^+$ such that $X \in U$ and $\Omega \in V$. Choose

a clopen neighbourhood C of X in $\mathfrak{I}_\kappa$ such that $C \subset U$ and note that C is also clopen in $\mathfrak{I}_\kappa^+$. Hence $\mathfrak{I}_\kappa^+$ is disconnected.

(4) $\Rightarrow$ (1). Assume that $\operatorname{ind} \mathfrak{I}_\kappa > 0$. Since τ_{w} is coarser than τ_d and zero-dimensional, the space $\mathfrak{I}_\kappa$ is totally disconnected. Let U be an open neighbourhood of Ω in $\mathfrak{I}_\kappa^+$ such that $V = \mathfrak{I}_\kappa \setminus U \neq \emptyset$. Let C be the component of Ω in U. Since Lemma 3.7.4 is true for T_1-spaces it suffices to show that C is not closed in the space $\mathfrak{I}_\kappa^+$. By Theorem 4.1.1, if $\kappa = \omega$, and Theorem 4.1.2, if $\kappa > \omega$, there is a $B \subset \kappa$ with $\varphi(B) = \infty$ and $\lim_{\alpha \in B} \varphi(\{\alpha\}) = 0$. Pick a set $X \in V$ and let $E = B \cup X$. Then E is a countably infinite set with $\varphi(E) = \infty$ and $\lim_{e \in E} \varphi(\{e\}) = 0$ since $X \in \mathrm{Exh}(\varphi)$. Put $U' = \{Y \in U : Y \subset E\} \cup \{\Omega\}$, $V' = V \cap \mathfrak{I}_E = \mathfrak{I}_E \setminus U'$, and let C' be the component of Ω in U'. Now U' is an open neighbourhood of Ω in $\mathfrak{I}_E \cup \{\Omega\}$ such that $X \in \mathfrak{I}_E \setminus U'$. According to Theorem 3.3.15 and the proof of Theorem 3.7.5 the closure of C' intersects V'. Since C' is a subset of C we see that C is not closed in $\mathfrak{I}_\kappa^+$ and the proof is complete. $\square$

4.5 Submeasures and nonseparable complete Erdős spaces

The purpose of this section, is to establish a link between nonseparable complete Erdős spaces and certain submeasures on κ. This is inspired by the following fact for LSC measures.

Proposition 4.5.1. *If the space $\mathcal{I}_\kappa$ is generated by an LSC measure φ on $\kappa \geq \omega$ and* $\operatorname{ind} \mathcal{I}_\kappa > 0$*, then $\mathcal{I}_\kappa$ is homeomorphic to $\mathfrak{E}_{\mathrm{c}} \times (\lambda_{\mathrm{D}})^\omega \times \kappa_{\mathrm{D}}$, with $\lambda = lw(\mathcal{I}_\kappa)$.*

Proof. As in Example 4.4.5, the ideal $\mathcal{I}_\kappa$ (with $\mathrm{Exh}(\varphi) = \mathrm{Fin}(\varphi)$) can be embedded in the Banach space ℓ^1_κ. Indeed, consider the function $h\colon \mathcal{I}_\kappa \to \ell^1_\kappa$ given by

$$\big(h(X)\big)_\alpha = \begin{cases} \varphi(\{\alpha\}), & \text{if } \alpha \in X; \\ 0, & \text{if } \alpha \notin X. \end{cases}$$

Just as in Example 4.4.5 it follows that $\varphi(X \triangle Y) = \|h(X) - h(Y)\|_1$, so h is clearly an embedding. Furthermore, we have $h(\mathcal{I}_\kappa) = \big(\prod_{\alpha\in\kappa} E_\alpha\big) \cap \ell^1_\kappa$, with $E_\alpha = \{0, \varphi(\{\alpha\})\}$ for every $\alpha \in \kappa$. Applying Theorem 4.2.7, we find $\mathcal{I}_\kappa \approx \mathfrak{E}_{\mathrm{c}} \times (\lambda_{\mathrm{D}})^\omega \times \kappa_{\mathrm{D}}$ with $\lambda = lw(\mathcal{I}_\kappa)$ and $\kappa = w(\mathcal{I}_\kappa)$ by Proposition 4.4.4. $\square$

In the proof of Lemma 4.5.3 we use the following general result.

Lemma 4.5.2. *Let (Z, τ) be a topological space and let I be a nonempty set. Suppose that $\varphi_i\colon Z \to [0, \infty]$ is an LSC function for every $i \in I$. Fix $M > 1$ and let $f_i\colon Z \to (1/M, M)$ be a continuous function for each $i \in I$. Define for $i \in I$ the function $\psi_i\colon Z \to [0, \infty]$ by $\psi_i = f_i \cdot \varphi_i$. Consider the functions $\varphi, \psi\colon Z \to [0, \infty]$ given by $\varphi = \sum_{i\in I} \varphi_i$ and $\psi = \sum_{i\in I} \psi_i$. Then $\psi^{-1}\big([0,\infty)\big) = \varphi^{-1}\big([0,\infty)\big)$ and $\tau_\varphi = \tau_\psi$.*

Proof. Clearly, $\varphi/M \leq \psi \leq M\varphi$, from which it follows immediately that $\psi^{-1}\big([0,\infty)\big) = \varphi^{-1}\big([0,\infty)\big)$.

We show that $\tau_\varphi = \tau_\psi$. We will use Lemma 4.3.10 to do this. It is not difficult to see that ψ_i is an LSC function for every $i \in I$. By a symmetry argument it is then enough to show that $\tau_\varphi \subset \tau_\psi$, that is, we have to prove that φ is continuous with respect to τ_ψ. We know from Lemma 4.3.10 that φ is an LSC function with respect to τ, so certainly

φ is an LSC function with respect to τ_ψ. It is therefore left to show that for all $t \in (0,\infty)$ the set $\varphi^{-1}\big([0,t)\big)$ is open in τ_ψ.

Take $t \in (0,\infty)$. Since $\psi^{-1}\big([0,\infty)\big) = \varphi^{-1}\big([0,\infty)\big)$ we know that $\varphi^{-1}\big([0,t)\big) \subset \psi^{-1}\big([0,\infty)\big)$. Clearly, φ_i is continuous with respect to τ_{ψ_i} for every $i \in I$. With Lemma 4.3.10 we know that $\tau_{\psi_i} \subset \tau_\psi$ on $\psi^{-1}\big([0,\infty)\big)$, so every φ_i is continuous with respect to τ_ψ on $\psi^{-1}\big([0,\infty)\big)$. Take $z \in \varphi^{-1}\big([0,t)\big)$ and put $\varepsilon = t - \varphi(z)$. We know that $\psi(z) < \infty$ so we can find a finite set $F \subset I$ such that $\sum_{i \in I\setminus F} \psi_i(z) < \varepsilon/(2M)$. Using that $\sum_{i\in F}\varphi_i$ is continuous with respect to τ_ψ on $\psi^{-1}\big([0,\infty)\big)$ and that $\psi^{-1}\big([0,\infty)\big) \in \tau_\psi$, we find an open set $U \subset \psi^{-1}\big([0,\infty)\big)$ in τ_ψ such that $z \in U$ and for all $z' \in U$ we have

$$\sum_{i\in F} \varphi_i(z') < \varphi(z) + \varepsilon/2.$$

Let $\tilde{\psi} = \sum_{i\in I\setminus F}\psi_i$, then we can write

$$\psi = \sum_{i\in F}\psi_i + \tilde{\psi}.$$

Lemma 4.3.10 tells us that $\sum_{i\in F}\psi_i$ and $\tilde{\psi}$ are LSC functions and that $\tau_{\tilde{\psi}} \subset \tau_\psi$ on $\psi^{-1}\big([0,\infty)\big)$. Define the set $V \subset \psi^{-1}\big([0,\infty)\big)$ by

$$V = U \cap \tilde{\psi}^{-1}\left(\left[0, \frac{\varepsilon}{2M}\right)\right).$$

It is clear that $z \in V \in \tau_\psi$. Moreover, for every $z' \in V$ we have

$$\begin{aligned} \varphi(z') &= \sum_{i\in F}\varphi_i(z') + \sum_{i\in I\setminus F}\varphi_i(z') \\ &\le \sum_{i\in F}\varphi_i(z') + M\,\tilde{\psi}(z') \\ &< \varphi(z) + \frac{\varepsilon}{2} + M\frac{\varepsilon}{2M} = t. \end{aligned}$$

We showed that $z \in V \subset \varphi^{-1}\big([0,t)\big)$ with $V \in \tau_\psi$, which means that $\varphi^{-1}\big([0,t)\big)$ is open in τ_ψ. $\square$

In the next lemma we look at submeasures that behave a bit like measures.

Lemma 4.5.3. *Let φ be a Kadec submeasure on $\kappa > \omega$ such that* $\mathrm{Exh}(\varphi) = \mathrm{Fin}(\varphi)$. *Suppose that there exists a partition $\{A_\beta : \beta \in \kappa\}$ of κ with $|A_\beta| \leq \omega$ for every $\beta \in \kappa$, $|\{\beta \in \kappa : \mathrm{ind}\, \mathcal{I}_{A_\beta} > 0\}| = \kappa$ and with the property that there exist an $M \in \mathbb{N}$ such that for every $X \in \mathrm{Exh}(\varphi)$ we have*

$$(*) \qquad \sum_{\beta\in\kappa} \varphi(X \cap A_\beta) \leq M\varphi(X).$$

Then $\mathcal{I}_\kappa$ is homeomorphic to $\mathfrak{E}_{\mathrm{c}} \times (\kappa_{\mathrm{D}})^\omega$.

Proof. Note that $(*)$ is trivially true for $X \notin \mathrm{Exh}(\varphi) = \mathrm{Fin}(\varphi)$. By taking appropriate unions of different sets A_β if necessary, and using the subadditivity of φ, one sees that we may, and we will from now on, assume that $\{A_\beta : \beta \in \kappa\}$ is a partition of κ such that $\mathrm{ind}\, \mathcal{I}_{A_\beta} > 0$ and $|A_\beta| = \omega$ for all $\beta \in \kappa$.

Claim 4.5.4. *For all $\beta \in \kappa$ the function φ_{A_β} is an L-Lelek function with compact domain for which $\varphi_{A_\beta}^{-1}(0) = \{\emptyset\}$.*

Proof. Pick a $\beta \in \kappa$. We only have to prove that φ_{A_β} is an L-Lelek function on 2^{A_β}, that is, that $G^\infty_{\varphi_{A_\beta}}$ is dense in $L^\infty_{\varphi_{A_\beta}}$ (see Definition 4.2.1). Let $X \subset A_\beta$ be arbitrary and consider a standard neighbourhood U of X in 2^{A_β}. So we have $U = \{Y \subset A_\beta : Y \cap F = X \cap F\}$ for some finite set $F \subset A_\beta$. Note that $\mathcal{I}_{A_\beta} \cap U$ is a clopen subspace of $\mathcal{I}_{A_\beta}$ that contains $X \cap F$. Let t be such that $\varphi_{A_\beta}(X \cap F) < t < \infty$. It now suffices to show the existence of some $Y \in \mathcal{I}_{A_\beta} \cap U$ with $\varphi_{A_\beta}(Y) = t$. We know that φ_{A_β} is a continuous function on $\mathcal{I}_{A_\beta}$. If there were no $Y \in \mathcal{I}_{A_\beta} \cap U$ with $\varphi_{A_\beta}(Y) = t$ then the nonempty clopen subset $\{Y \in \mathcal{I}_{A_\beta} \cap U : \varphi_{A_\beta}(Y) < t\} \subset \mathcal{I}_{A_\beta}$ would be φ-bounded. This violates Theorem 4.1.1 and thereby the claim is proved. $\diamondsuit$

Put $T = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ and define the function $\psi_{A_\beta} : T^{A_\beta} \to [0, \infty]$ by $\psi_{A_\beta}(x) = \sum_{\alpha\in A_\beta} x_\alpha$. Note that ψ_{A_β} coincides with the $\ell^1_{A_\beta}$-norm when both are restricted to the set $\{x \in T^{A_\beta} : \sum_{\alpha\in A_\beta} x_\alpha < \infty\}$. We can now apply Lemma 4.2.2 to $\varphi_{A_\beta} : 2^{A_\beta} \to [0, \infty]$ and ψ_{A_β} for every $\beta \in \kappa$, to find a homeomorphism $h_\beta : 2^{A_\beta} \to T^{A_\beta}$ and a continuous function $f_\beta : 2^{A_\beta} \to (1/2, 2)$ such that $\psi_{A_\beta}(h_\beta(X)) = f_\beta(X)\varphi_{A_\beta}(X)$ for every $X \in 2^{A_\beta}$. Define $\psi : 2^\kappa \to [0,\infty]$ by

$$\psi(X) = \sum_{\beta\in\kappa} \psi_{A_\beta}(h_\beta(X \cap A_\beta)),$$

so that

$$\psi(X) = \sum_{\beta\in\kappa} f_\beta(X \cap A_\beta)\varphi_{A_\beta}(X \cap A_\beta) = \sum_{\beta\in\kappa} \tilde{f}_\beta(X)\varphi_\beta(X),$$

where $\tilde{f}_\beta : 2^\kappa \to (1/2, 2)$ is given by $\tilde{f}_\beta(X) = f_\beta(X \cap A_\beta)$ and $\varphi_\beta : 2^\kappa \to [0,\infty]$ is given by $\varphi_\beta(X) = \varphi_{A_\beta}(X \cap A_\beta)$. Since the function $X \mapsto X \cap A_\beta$ from 2^κ to 2^{A_β} is continuous it follows that $\tilde{f}_\beta$ is continuous for all $\beta \in \kappa$ and φ_β is an LSC function for all $\beta \in \kappa$. Observe that ψ is not in general a submeasure on κ.

Now consider the function $\tilde{\varphi}\colon 2^\kappa \to [0,\infty]$ given by $\tilde{\varphi} = \sum_{\beta\in\kappa} \varphi_\beta$. Since φ is a Kadec submeasure it is clear that every φ_{A_β} is a Kadec submeasure and we can apply Corollary 4.3.11 to see that $\tilde{\varphi}$ is a Kadec submeasure as well. Using that φ is an LSC submeasure it is not difficult to show that $\varphi \le \tilde{\varphi}$. By assumption we have $\tilde{\varphi} \le M\varphi$ and these inequalities, together with the equality $\mathrm{Exh}(\varphi) = \mathrm{Fin}(\varphi)$, imply that $\mathrm{Exh}(\tilde{\varphi}) = \mathrm{Fin}(\tilde{\varphi})$ with $\mathrm{Exh}(\tilde{\varphi}) = \mathrm{Exh}(\varphi)$. We also see that the metrics d_φ and $d_{\tilde{\varphi}}$ on $\mathrm{Exh}(\varphi)$ as defined in Theorem 4.3.2 are uniformly equivalent. We can now write

$$\mathcal{I}_\kappa = (\mathrm{Exh}(\varphi), \tau_{d_\varphi}) = (\mathrm{Exh}(\tilde{\varphi}), \tau_{d_{\tilde{\varphi}}}) = (\mathrm{Exh}(\tilde{\varphi}), \tau_{\tilde{\varphi}}).$$

Since $\psi = \sum_{\beta\in\kappa} \tilde{f}_\beta \cdot \varphi_\beta$ it follows from Lemma 4.5.2 that $\mathrm{Fin}(\tilde{\varphi}) = \psi^{-1}\big([0,\infty)\big)$ and $\tau_{\tilde{\varphi}} = \tau_\psi$. Using that $\mathrm{Exh}(\tilde{\varphi}) = \mathrm{Fin}(\tilde{\varphi})$ we get with the previous equalities that

$$\mathcal{I}_\kappa = (\psi^{-1}\big([0,\infty)\big), \tau_\psi).$$

Next, define $H : 2^\kappa \to T^\kappa$ coordinatewise by $H(X)_\alpha = h_\beta(X \cap A_\beta)_\alpha$, whenever $\alpha \in A_\beta$. This is obviously a homeomorphism with respect to the underlying product topologies. We now find

$$\psi(X) = \sum_{\alpha\in\kappa} H(X)_\alpha,$$

hence $X \mapsto \sum_{\alpha\in\kappa} H(X)_\alpha$ is continuous on $\mathfrak{I}_\kappa$. Note that whenever $\psi(X) < \infty$, this value equals the ℓ^1_κ-norm of $H(X)$. Using Remark 4.2.5 we find that the map $H : \mathfrak{I}_\kappa \to T^\kappa \cap \ell^1_\kappa$ is continuous. A symmetric argument yields continuity of its inverse and combining this with the equality $T^\kappa \cap \ell^1_\kappa = \mathfrak{E}^1_{\mathrm{c}}(\kappa,\kappa)$ and Theorem 4.2.8 we conclude that $\mathfrak{I}_\kappa \approx \mathfrak{E}_{\mathrm{c}} \times (\kappa_D)^\omega$. □

Theorem 4.5.5. *Let φ be an LSC submeasure on $\kappa > \omega$ such that* $\mathrm{Exh}(\varphi) = \mathrm{Fin}(\varphi)$ *and $lw(\mathfrak{I}_\kappa) = \lambda > \omega$. Suppose that there exists a partition $\{A_\beta : \beta \in \kappa\}$ of κ with $|A_\beta| \le \omega$ for every $\beta \in \kappa$ and with the property that there exists an $M \in \mathbb{N}$ such that for every $X \in \mathrm{Exh}(\varphi)$ we have*

$$\sum_{\beta\in\kappa} \varphi(X \cap A_\beta) \le M\varphi(X).$$

Then $\mathfrak{I}_\kappa$ is homeomorphic to $\mathfrak{E}_{\mathrm{c}} \times (\lambda_{\mathrm{D}})^\omega \times \kappa_{\mathrm{D}}$.

Proof. We start with the observation that we may assume without loss of generality that φ is a Kadec submeasure on κ. We can see this as follows. Apply Theorem 4.3.12 to find a Kadec submeasure ψ_β on A_β such that $\varphi_{A_\beta} \le \psi_\beta \le 2\varphi_{A_\beta}$. Define $\psi\colon 2^\kappa \to [0,\infty]$ by

$$\psi(X) = \sum_{\beta\in\kappa} \psi_\beta(X \cap A_\beta).$$

By Corollary 4.3.11 ψ is a Kadec submeasure on κ. Using that φ is an LSC submeasure and $\varphi_{A_\beta} \le \psi_\beta$ for all $\beta \in \kappa$ it follows easily that $\varphi \le \psi$. Furthermore, since $\psi_\beta \le 2\varphi_{A_\beta}$ for all $\beta \in \kappa$ we also have that $\psi \le 2M\varphi$ by assumption. Together with the fact that $\mathrm{Exh}(\varphi) = \mathrm{Fin}(\varphi)$ this implies that $\mathrm{Exh}(\psi) = \mathrm{Fin}(\psi)$, with $\mathrm{Exh}(\psi) = \mathrm{Exh}(\varphi)$, and also that the metrics d_φ and d_ψ on $\mathrm{Exh}(\varphi)$ as defined in Theorem 4.3.2 are uniformly equivalent. So ψ produces the same space $\mathfrak{I}_\kappa$ as φ. In addition, it is clear that $\sum_{\beta\in\kappa} \psi(X \cap A_\beta) = \psi(X)$ for all $X \subset \kappa$ so ψ satisfies the conditions of this theorem. We see that we can replace φ by ψ and therefore we will continue the proof assuming that φ is a Kadec submeasure on κ.

Using Proposition 4.4.4 we take an $n_0 \in \mathbb{N}$ such that the cardinality of the set $C = \{\alpha \in \kappa : \varphi(\{\alpha\}) < 1/n_0\}$ is equal to λ. Using transfinite recursion we construct a collection $\{A'_\beta : \beta \in \lambda\}$ of pairwise disjoint countable subsets of κ where each set A'_β is a union of sets A_γ with $\gamma \in \kappa$, such that $\operatorname{ind} \mathcal{I}_{A'_\beta} > 0$ for all $\beta \in \lambda$. Assume that $\alpha \in \lambda$ is such that A'_β has been found for $\beta < \alpha$. Put $B = \bigcup_{\beta<\alpha} A'_\beta$ and note that $|B| < \lambda$. It follows from Proposition 4.4.4 that $lw(\mathcal{I}_{C\setminus B}) = \lambda$ and since clearly $\operatorname{Exh}(\varphi_{C\setminus B}) = \operatorname{Fin}(\varphi_{C\setminus B})$ we get from Proposition 4.4.6 and Remark 4.4.1 that there is a countable set $C_\alpha \subset C \setminus B$ such that $\operatorname{ind} \mathcal{I}_{C_\alpha} > 0$. Put $A'_\alpha = \bigcup\{A_\beta : A_\beta \cap C_\alpha \neq \emptyset\}$ and note that it is countable and disjoint from A'_β for $\beta < \alpha$. In this way we get the desired sets A'_β for every $\beta \in \lambda$.

If $\lambda = \kappa$, then put $A' = \bigcup_{\beta\in\lambda} A'_\beta$ and define the collection $\mathcal{A}$ of subsets of κ as

$$\mathcal{A} = \{A'_\beta : \beta \in \lambda\} \cup \{A_\beta : A_\beta \cap A' = \emptyset\}.$$

With the subadditivity of φ and the fact that every A'_β is a union of sets A_γ with $\gamma \in \kappa$, it follows easily that $\mathcal{A}$ is a partition of κ that satisfies the conditions of Lemma 4.5.3, which means that $\mathcal{I}_\kappa \approx \mathfrak{E}_{\mathrm{c}} \times (\kappa_{\mathrm{D}})^\omega$.

Now suppose that $\lambda < \kappa$. Again, put $A' = \bigcup_{\beta\in\lambda} A'_\beta$ and now define the collection $\mathcal{A}$ of subsets of κ as

$$\mathcal{A} = \{A'_\beta : \beta \in \lambda\} \cup \{A_\beta : A_\beta \cap (C \setminus A') \neq \emptyset\}.$$

We have $|\mathcal{A}| = \lambda$, so we can write $\mathcal{A} = \{B_\beta : \beta \in \lambda\}$. We define $A = \bigcup \mathcal{A}$, so $\mathcal{A}$ is a partition of A. Since every element of $\mathcal{A}$ is at most countable we have $|A| = \lambda$. We know that $\operatorname{ind} A'_\beta > 0$ for all $\beta \in \lambda$, which means that $|\{\beta \in \lambda : \operatorname{ind} \mathcal{I}_{B_\beta} > 0\}| = \lambda$. Furthermore, since φ is subadditive and every B_β is a union of sets A_γ with $\gamma \in \kappa$, we have for every $X \subset A$ that

$$\sum_{\beta\in\lambda} \varphi_A(X \cap B_\beta) \leq \sum_{\beta\in\kappa} \varphi(X \cap A_\beta) \leq M\varphi(X) = M\varphi_A(X).$$

Note that $\operatorname{Exh}(\varphi_A) = \operatorname{Fin}(\varphi_A)$ and since it is clear that φ_A is a Kadec submeasure on A we may apply Lemma 4.5.3, which says that $\mathcal{I}_A$ is homeomorphic to $\mathfrak{E}_{\mathrm{c}} \times (\lambda_{\mathrm{D}})^\omega$.

Note that $A \supset C$, so if $\alpha \in \kappa \setminus A$ then $\varphi(\{\alpha\}) \geq 1/n_0$ and this means that $\mathcal{I}_{\kappa\setminus A}$ is the collection of all finite subsets of $\kappa \setminus A$ with the discrete topology. Since $|\mathcal{I}_{\kappa\setminus A}| = \kappa$, we see that $\mathcal{I}_{\kappa\setminus A}$ is homeomorphic to κ_{D}. Clearly, $\mathcal{I}_\kappa$ is homeomorphic to $\mathcal{I}_A \times \mathcal{I}_{\kappa\setminus A}$, which gives the desired result. $\square$

Remark 4.5.6. Let φ be an LSC submeasure on $\kappa > \omega$ such that $\mathrm{Exh}(\varphi) = \mathrm{Fin}(\varphi)$ and suppose that there exists a partition $\{A_\beta : \beta \in \kappa\}$ of κ with $|A_\beta| \leq \omega$ for every $\beta \in \kappa$ and with the property that there is an $M \in \mathbb{N}$ such that for every $X \in \mathrm{Exh}(\varphi)$ we have

$$\sum_{\beta\in\kappa} \varphi(X \cap A_\beta) \leq M\varphi(X).$$

Let λ be the local weight of $\mathcal{I}_\kappa$. Then it follows easily from Proposition 4.4.7, together with [10, Remark 13], and Theorem 4.5.5 that

$$\mathrm{ind}\, \mathcal{I}_\kappa > 0 \Longleftrightarrow \mathcal{I}_\kappa \approx \mathfrak{E}_{\mathrm{c}} \times (\lambda_{\mathrm{D}})^\omega \times \kappa_{\mathrm{D}}.$$

This means that for this kind of submeasures we can add the statement that $\mathcal{I}_\kappa$ is homeomorphic to $\mathfrak{E}_{\mathrm{c}} \times (\lambda_{\mathrm{D}})^\omega \times \kappa_{\mathrm{D}}$ to the list of equivalent statements of Theorem 4.1.2.

Example 4.5.7. Until Lemma 4.5.3, we were only able to classify (up to homeomorphism) the zero-dimensional ideals of Theorem 4.1.2, the ideals mentioned in Corollary 4.4.7, and the ideals generated by LSC measures mentioned in Proposition 4.5.1. Let us now consider the submeasure φ on $\mathbb{N} \times \kappa$ for some $\kappa > \omega$ given by

$$\varphi(X) = \sum_{\alpha\in\kappa} \left(|\pi_\alpha(X) \cap I_0| + \sum_{k=1}^{\infty} \frac{\min(k, |\pi_\alpha(X) \cap I_k|)}{k^2} \right),$$

where $\pi_\alpha(X) = \{n : (n, \alpha) \in X\}$ and $I_k = [2^k, 2^{k+1}) \cap \mathbb{N}$, for $k \in \omega$. This submeasure restricted to $\mathbb{N} \times \{\alpha_0\}$ for any $\alpha_0 \in \kappa$ is in essence the submeasure studied in [21, Example 1.11.1]. In this example, Farah proves that there does not exist a measure on $\mathbb{N}$ generating the same ideal as this submeasure. Note that Theorem 4.5.5 applies to φ and if $\mathcal{I}_\kappa$ were generated by a measure on $\mathbb{N} \times \kappa$, this would imply that the ideal in Farah's example is generated by a measure as well. Hence, we really need Theorem 4.5.5 to conclude that $\mathcal{I}_\kappa \approx \mathfrak{E}_{\mathrm{c}} \times (\kappa_{\mathrm{D}})^\omega$.

We finish with some questions, of which the first is related to Remark 4.4.2.

Question 4.5.8. Is it true that $\operatorname{ind} \mathcal{I}_\kappa \leq 1$ for any space $\mathcal{I}_\kappa$ that is generated by an LSC submeasure φ on $\kappa > \omega$ with $\operatorname{Exh}(\varphi) = \operatorname{Fin}(\varphi)$?

Question 4.5.9. Let φ be an LSC submeasure on $\kappa > \omega$ such that $\operatorname{Exh}(\varphi) = \operatorname{Fin}(\varphi)$. Can we always find a partition $\{A_\beta : \beta \in \kappa\}$ of κ as mentioned in Theorem 4.5.5?

With the first part of the proof of Theorem 4.5.5 we see that a positive answer to this question would solve Question 4.3.13 in the affirmative for this special class of submeasures φ. With Remark 4.4.2 we could then also answer Question 4.5.8. Furthermore, with Remark 4.5.6 we would have a negative answer to our following main question in this case.

Question 4.5.10. Does there exist an LSC submeasure φ on some cardinal number $\kappa > \omega$ such that $\operatorname{Exh}(\varphi) = \operatorname{Fin}(\varphi)$ and $\operatorname{ind} \mathcal{I}_\kappa > 0$, where $\mathcal{I}_\kappa$ is not homeomorphic to a space of the form $\mathfrak{E}_\mathrm{c} \times (\lambda_\mathrm{D})^\omega \times \kappa_\mathrm{D}$?

We note that if we drop the assumption that $\operatorname{Exh}(\varphi) = \operatorname{Fin}(\varphi)$ we can answer this question in the affirmative. Consider for example the LSC submeasure ψ in Remark 4.4.3 with the property that $(\operatorname{Exh}(\psi), \tau_d)$ is homeomorphic to $\mathfrak{E}_\mathrm{c}^\omega \times \kappa_\mathrm{D}$. According to Dijkstra and Valkenburg [14, Theorem 14] this space is nonhomeomorphic to any nonseparable complete Erdős space.

Bibliography

[1] M. Abry and J. J. Dijkstra, *On topological Kadec norms*, Math. Ann. **332** (2005), 759–765.

[2] M. Abry, J. J. Dijkstra and J. van Mill, *On one-point connectifications*, Topology Appl. **154** (2007), 725–733.

[3] C. Bessaga and A. Pełczyński, *Selected topics in infinite-dimensional topology*, PWN—Polish Scientific Publishers, Warsaw (1975), monografie Matematyczne, Tom 58. [Mathematical Monographs, Vol. 58].

[4] J. W. Cannon, *A positional characterization of the* $(n-1)$*-dimensional Sierpiński curve in* $S^n (n \neq 4)$, Fund. Math. **79** (1973), 107–112.

[5] J. J. Dijkstra, *A criterion for Erdős spaces*, Proc. Edinb. Math. Soc. (2) **48** (2005), 595–601.

[6] J. J. Dijkstra, *On homeomorphism groups and the compact-open topology*, Amer. Math. Monthly **112** (2005), 910–912.

[7] J. J. Dijkstra, *On homeomorphism groups of Menger continua*, Trans. Amer. Math. Soc. **357** (2005), 2665–2679.

[8] J. J. Dijkstra, *Characterizing stable complete Erdős space*, preprint.

[9] J. J. Dijkstra and J. van Mill, *Homeomorphism groups of manifolds and Erdős space*, Electron. Res. Announc. Amer. Math. Soc. **10** (2004), 29–38 (electronic).

[10] J. J. Dijkstra and J. van Mill, *Characterizing complete Erdős space*, Canad. J. Math. **61** (2009), 124–140.

[11] J. J. Dijkstra and J. van Mill, *Erdős space and homeomorphism groups of manifolds*, Mem. Amer. Math. Soc., in press.

[12] J. J. Dijkstra, J. van Mill and J. Steprāns, *Complete Erdős space is unstable*, Math. Proc. Cambridge Philos. Soc. **137** (2004), 465–473.

[13] J. J. Dijkstra, J. van Mill and K. I. S. Valkenburg, *On nonseparable Erdős spaces*, J. Math. Soc. Japan **60** (2008), 793–818.

[14] J. J. Dijkstra and K. I. S. Valkenburg, *The instability of nonseparable complete Erdős spaces and representations in $\mathbb{R}$-trees*, prepint.

[15] J. J. Dijkstra, K. I. S. Valkenburg and D. Visser, *Nonseparable complete Erdős spaces and submeasures on uncountable cardinals*, preprint.

[16] J. J. Dijkstra and D. Visser, *On generalized Erdős spaces*, Topology Appl. **155** (2008), 233–251.

[17] J. J. Dijkstra and D. Visser, *Homeomorphism groups of Sierpiński carpets and Erdős space*, preprint.

[18] R. Engelking, *Dimension theory*, North-Holland Publishing Co., Amsterdam (1978).

[19] R. Engelking, *General topology*, *Sigma Series in Pure Mathematics*, vol. 6, Heldermann Verlag, Berlin, second ed. (1989).

[20] P. Erdős, *The dimension of the rational points in Hilbert space*, Ann. of Math. (2) **41** (1940), 734–736.

[21] I. Farah, *Analytic quotients: theory of liftings for quotients over analytic ideals on the integers*, Mem. Amer. Math. Soc. **148** (2000), xvi+177.

[22] A. S. Kechris, *Classical descriptive set theory*, *Graduate Texts in Mathematics*, vol. 156, Springer-Verlag, New York (1995).

[23] M. Levin and R. Pol, *A metric condition which implies dimension* ≤ 1, Proc. Amer. Math. Soc. **125** (1997), 269–273.

[24] J. van Mill, *The infinite-dimensional topology of function spaces*, North-Holland Publishing Co., Amsterdam (2001).

[25] J. R. Munkres, *Elements of algebraic topology*, Addison-Wesley Publishing Company, Menlo Park, CA (1984).

[26] L. G. Oversteegen and E. D. Tymchatyn, *On the dimension of certain totally disconnected spaces*, Proc. Amer. Math. Soc. **122** (1994), 885–891.

[27] W. Sierpiński, *Sur une définition topologique des ensembles* $F_{\sigma\delta}$, Fund. Math. **6** (1924), 24–29.

[28] S. Solecki, *Analytic ideals*, Bull. Symbolic Logic **2** (1996), 339–348.

[29] S. Solecki, *Analytic ideals and their applications*, Ann. Pure Appl. Logic **99** (1999), 51–72.

[30] G. T. Whyburn, *Topological characterization of the Sierpiński curve*, Fund. Math. **45** (1958), 320–324.

Notation

ω, 1	The set of natural numbers, including zero
$\mathbb{N}$, 1	The set of natural numbers starting from 1
$\mathbb{Z}$, 1	The set of integers
$\mathbb{Q}$, 1	The set of rational numbers
$\mathbb{R}$, 1	The set of real numbers
$\mathrm{Int}(A)$, 1	The interior of the set A
$\overline{A}$, 1	The closure of the set A
∂A, 1	The boundary of the set A
$\operatorname{diam} A$, 2	The diameter of the set A
$\mathcal{H}(X)$, 2	The autohomeomorphism group of X
$[A, B]$, 2	The set of all homeomorphisms that map A into B
$X \approx Y$, 3	The space X is homeomorphic to the space Y
$\mathcal{P}(A)$, 3	The powerset of A
$\emptyset$, 3	The empty set
$\mathrm{Exh}(\varphi)$, 4	The exhaustive ideal of a submeasure φ
$\mathrm{Fin}(\varphi)$, 4	The finite ideal of a submeasure φ
$\lvert A \rvert$, 4	The cardinality of the set A
$\mathfrak{c}$, 5	The cardinality of $\mathbb{R}$
$A \triangle B$, 5	The symmetric difference of the sets A and B
$\hat{\mathbb{R}}$, 5	The compactification $[-\infty, \infty]$ of $\mathbb{R}$
LSC, 6	Lower semi-continuous
$f \restriction V$, 6	The restriction of the function f to V
dim, 8	The covering dimension
ind, 8	The small inductive dimension
Ind, 8	The large inductive dimension
$\mathfrak{E}$, 9	Erdős space

$w\,(Z)$, 86	The weight of a space Z
$lw(Z)$, 86	The local weight of a space Z
ℓ^p_κ, 86	A generalization of ℓ^p, constructed in $\mathbb{R}^\kappa$
$\mathfrak{E}^1_{\mathrm{c}}(\lambda,\kappa)$, 87	A generalization of $\mathfrak{E}_{\mathrm{c}}$, constructed in $\mathbb{R}^\kappa$
$\mathfrak{I}_A$, 89	The metric space $(\mathrm{Exh}(\varphi),\tau_d)$ for some LSC submeasure φ on a set A
τ_f, 89	The weakest topology on the domain of a map f that makes f continuous and that contains a given topology on the domain of f
$X\bigoplus Y$, 94	The topological sum of the spaces X and Y

Index of Names

Subject Index

Italicized page numbers refer to pages where the definition of the corresponding entry can be found. Page numbers in normal type indicate a textual reference.

Samenvatting

Representaties van Erdősruimten door middel van homeomorfismegroepen en halfcontinue functies op productruimten

De twee ruimten die de kern van dit proefschrift vormen zijn de *Erdősruimte*, die we weergeven met $\mathfrak{E}$, en de *volledige Erdősruimte*, die we weergeven met $\mathfrak{E}_c$. Beide ruimten zijn in 1940 door Paul Erdős geïntroduceerd. Hij bewees dat $\mathfrak{E}$ en $\mathfrak{E}_c$ totaal onsamenhangende, eendimensionale ruimten zijn. Het is bovendien niet moeilijk om in te zien dat $\mathfrak{E}$ en $\mathfrak{E}_c$ homeomorf zijn met hun kwadraat. Dit betekent dat $\dim \mathfrak{E} = \dim \mathfrak{E}^2 = 1$, een eigenschap die deze ruimte tot een belangrijk voorbeeld in de dimensietheorie maakt. Hetzelfde geldt natuurlijk voor $\mathfrak{E}_c$. Bij het lezen van dit proefschrift zal het duidelijk worden dat deze ruimten geen curiositeiten zijn, maar in verschillende situaties opduiken. In Hoofdstuk 2 bijvoorbeeld zegt onze hoofdstelling dat zekere homeomorfismegroepen van n-dimensionale Sierpiński tapijten voor $n \neq 3$, homeomorf zijn met $\mathfrak{E}$. Al met al hebben we flink wat krachtige wiskundige resultaten nodig om deze stelling te bewijzen. Dit resultaat kan dan ook als de hoofdstelling van het proefschrift gezien worden.

In Hoofdstuk 3 introduceren we gegeneraliseerde Erdősruimten. De hoofdstelling in dit hoofdstuk generaliseert een resultaat van Dijkstra, over Erdősachtige deelruimten van de Banachruimte ℓ^p, en een resultaat van Dijkstra en Van Mill over verpoolsbare idealen op de natuurlijke getallen. De ruimte die we in deze stelling bestuderen is een gegeneraliseerde Erdősruimte: het is een generalisatie van de constructie van de Erdősachtige deelruimten van ℓ^p en de verpoolsbare idealen bestudeert door Dijkstra en Van Mill.

In Hoofdstuk 4 houden we ons bezig met niet-separabele ruimten. De motivatie hiervoor is een resultaat van Dijkstra, Van Mill en Valkenburg. Zij hebben een generalisatie van Dijkstra's stelling over Erdősachtige deelruimten van ℓ^p gevonden door het domein van de normfunctie uit te breiden tot een overaftelbaar product van de reëele rechte. Met behulp van deze generalisatie en topologische karakterisaties van $\mathfrak{E}_c$ door Dijkstra en Van Mill, hebben zij gekarakteriseerd wanneer de resulterende, mogelijk niet-separabele, Erdősachtige ruimten homeomorf zijn met een zogenoemde niet-separabele volledige Erdősruimte. Geïnspireerd door deze resultaten bewijzen we een uitbreiding van de stelling van Dijkstra en Van Mill over verpoolsbare idealen voor submaten op overaftelbare kardinaalgetallen. We zijn nu in het bijzonder geïnteresseerd in de vraag wanneer de door deze submaten gegenereerde idealen homeomorf zijn met een niet-separabele volledige Erdősruimte. In de laatste sectie van Hoofdstuk 4 presenteren we een gedeeltelijk antwoord op deze vraag door te laten zien dat voor een speciale klasse van submaten de desbetreffende idealen homeomorf zijn met een niet-separabele volledige Erdősruimte dan en slechts dan als de kleine inductieve dimensie van deze idealen groter dan nul is.

In Hoofdstuk 1 behandelen we de basistheorie die nodig is voor de latere hoofdstukken.